Born's Reciprocity Principle in Stochastic Phase Space (*).

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In 1938 M. BORN formulated (1) the principle of reciprocity and adopted it as the basis of his attempt to achieve a consistent unification of relativity and quantum theory. According to this principle (cf. ref. (2), p. 463) "the laws of nature are symmetrical with regard to space-time and momentum-energy", and in particular, they are invariant under the transformation

(1) \[ q^r \mapsto p^r, \quad p^r \mapsto -q^r. \]

Soon afterwards, BORN (3) as well as LANDÉ (4), postulated that in addition to obeying the basis equation

(2) \[ (k^0)^2 - \mathbf{k}^2 = m^2 c^2 \]

for 4-momentum, a quantum particle should possess a proper wave function (5,6) that describes its spatial-extension properties in its rest frame and should be characterized by a length \( a \) associated (cf. ref. (5), p. 141) with that particle in such a manner that the reciprocal counterpart of (2), namely

(3) \[ (x^0)^2 - x^2 = a^2, \]

is also satisfied (cf. ref. (2) for a review).

In the stochastic phase-space approach to quantum mechanics (6,8) the concept of proper wave function for a quantum particle arises naturally when considering phase
BORN’S RECIPROCITY PRINCIPLE IN STOCHASTIC PHASE SPACE

space representations of the canonical commutation relations, since it mathematically corresponds to resolution generators $\xi$ and $\eta$ for irreducible representations of the Galilei and Poincaré groups, respectively, on $I^4$ spaces over the appropriate phase space (cf. refs. (8,9) for reviews). In the relativistic context, for massive spin-zero particles $\eta$ has the general form in ref. (6), eq. (B11) or ref. (9), eq. (3.24), which for real and rotationally invariant $\hat{\eta}(k)$ is equivalent to

$$\eta(q, p) = \int_{\mathbb{R}^4_+} \exp \left( -\frac{i}{\hbar} k \cdot q \right) \hat{\eta}(k \cdot p) \hat{\eta}(mck^0) \delta(k^2 - m^2 c^2) \, d^4k ,$$

where the integration extends over the positive-mass hyperboloid. The condition that in addition to being rotationally invariant $\hat{\eta}$ should be also be real follows (10,11) from the demand that $\eta$ give rise to a covariant and conserved probability current (whose nonrelativistic counterpart merges (11) into the conventional current in the limit of pointlike particles). However, the range of a priori available choices for $\hat{\eta}$ cannot be narrowed down any further without invoking some additional principle, such as the intuitively very appealing reciprocity principle by BORN.

In accordance with this reciprocity principle, $\eta(q, p)$ should satisfy (for all $p \in \mathbb{R}^4_+$) the Klein-Gordon equation

$$\left( \frac{\partial}{\partial q^\nu} \frac{\partial}{\partial q^\nu} - \frac{m^2 c^2}{\hbar^2} \right) \eta(p, q) = 0 ,$$

that follows from (2), as well as (for all $q$ in Minkowski space) the Born-Landé equation

$$\left( \frac{\partial}{\partial p^r} \frac{\partial}{\partial p^r} - \frac{\alpha^2}{\hbar^2} \right) \eta(q, p) = 0$$

that follows from (3)—both by the usual substitutions

$$k^r \mapsto i\hbar \frac{\partial}{\partial q^r} , \quad x^r \mapsto i\hbar \frac{\partial}{\partial p^r} .$$

Now, (5) is one immediate consequence of (4) by virtue of (2), but when we insert (4) into (6) we obtain a new restriction on $\hat{\eta}$, namely the equation

$$\hat{\eta}' - \left( \frac{\alpha^2}{\hbar^2} \right) \hat{\eta} = 0 , \quad \hat{\eta}' = \frac{d}{dk^0} \hat{\eta}(mck^0) .$$

Since $\hat{\eta}(k) = \hat{\eta}(mck^0)$ has to be square integrable over $\mathbb{R}^4_+$, the only acceptable solution of (8) is

$$\hat{\eta}_a(mck^0) = Z_{m,a} \exp \left[ -\frac{\alpha}{\hbar} k^0 \right] ,$$