Lagrangian for a System of Charged Particles to Higher-Order Terms.

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(ricevuto l'11 Giugno 1979)

We give the complete Lagrangian for any number of charged particles interacting with each other up to the fourth-order terms assuming that there is no dipole radiation of the closed system.

Let the charges be the sources of the electromagnetic field. The whole system, using the Einstein-Infeld-Hoffman method (1) can be described by a Lagrangian (2)

\[ \mathcal{L} = \mathcal{L}^{(0)} + \frac{1}{c^2} \mathcal{L}^{(2)} + O(c^{-5}) , \]

where the radiation of electromagnetic waves occurs in the third-order terms in \( c \) (2). This radiation is determined by the second derivative of the dipole moment of the system, which is defined as

\[ d = \sum_{i=1}^{n} q_i r_i = \sum_{i=1}^{n} \frac{q_i}{m_i} m_i r_i = \text{const} \sum_{i=1}^{n} m_i r_i = \text{const} R \sum_{i=1}^{n} m_i , \]

where \( R \) is the position vector of the centre of inertia of the system. In any closed system the acceleration of the centre of inertia is zero, i.e. \( \ddot{R} = 0 \), hence from eq. (2) it follows \( \ddot{d} = 0 \). (The velocities are small, \( v \ll c \), so that nonrelativistic mechanics is applicable.) Therefore the system cannot radiate by dipole radiation. Further, if the ratio of charge to mass of all particles is the same then the system can be described by an approximate Lagrangian (of the same form as in the pure gravitational case (3,4))

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(2) C. G. Darwin: Philos. Mag., 39, 537 (1920).
so that all the well-known theorems of general dynamics may be made applicable \(^{(1,2)}\), \(i.e.\)

\[
\mathcal{L} = \mathcal{L}^{(0)} + \frac{1}{e^2} \mathcal{L}^{(2)} + \frac{1}{e^4} \mathcal{L}^{(4)} + O(e^{-5}),
\]

where in eqs. \((2)\) and \((3)\) \(\mathcal{L}^{(0)}\) are the zeroth-order terms, \(\mathcal{L}^{(2)}\) are the second-order terms, \(\mathcal{L}^{(4)}\) are the fourth-order terms and \(O(e^{-5})\) means the fifth-order terms. Now going to the third-order terms we have \(\mathcal{L}^{(3)} = 0\), since there is no dipole radiation \(^{(2)}\).

A particle with charge \(q_i\) and mass \(m_i\) travelling in an electromagnetic field has the following Lagrangian \(^{(2,5,6)}\):

\[
\mathcal{L} = -m_i c^2 \sqrt{1 - \frac{v_i^2}{c^2}} - q_i \varphi(r_i, t) + \frac{q_i}{c} A(r_i, t) \cdot v_i,
\]

where \(\varphi\) and \(A\) are the scalar and the vector potentials of the field. So the equations of motion can be derived from the Lagrangian \((4)\) \(^{(2)}\).

We suppose that we have a system of particles with the vector positions \(r_i(t)\) and charges \(q_i\), then the charge and the current density functions can be represented, respectively, as

\[
q(r, t) = \sum_{i=1}^{n} q_i \delta(r - r_i(t)), \quad j(r, t) = \sum_{i=1}^{n} q_i v_i \delta(r - r_i(t)),
\]

where \(\delta\) denotes the Dirac \(\delta\)-function \(^{(7)}\).

Evidently, we shall have

\[
\frac{\hat{c} \theta}{c t} + \nabla \cdot j = 0.
\]

Thus, we arrive at the equation of continuity.

Moreover, we can define

\[
\int_{V} |r - r_j|^\lambda \sum_{i=1}^{n} q_i \delta(r - r_i) \, d\mathbf{r} = \sum_{i=1}^{n} q_i |r_i - r_j|^\lambda, \quad \lambda = 0, \pm 1, \pm 2, \ldots,
\]

where all the charges are inside the volume \(V\) \(^{(7)}\).

It is easy to convince oneself that the total Lagrangian for a closed system consisting of \(n\) charged particles has the form \(^{(4,6,8)}\)

\[
\mathcal{L} = -\sum_{i=1}^{n} m_i c^2 \sqrt{1 - \frac{v_i^2}{c^2}} - \frac{1}{2} \sum_{i=1}^{n} q_i \varphi(r_i, t) + \frac{1}{2} \sum_{i=1}^{n} q_i v_i \cdot A(r_i, t),
\]

\(^{(4)}\) D. D. DIONYSIOU and D. A. VAIPOULOS: paper under publication.