Solutions on Manifolds.

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Summary. - First-order partial differential equations are solved which do not satisfy integrability conditions. Important examples of such equations in mathematical physics are the Liouville, the Dirac, the Vlasov equations etc. The set of boundary functions together with the vector field components, \( u_i(x), \ i = 4, 5, 6 \), determine the manifold, \( X_6 \), which gives the support of the distribution function. The solution is presented here for the case of the Vlasov equation. The main result in this paper is eq. (9). The proof of construction is not given here. This method can be used to study also stability problems. The nonexistence of integrability conditions manifests itself by the reduction of \( R^7 \) to the manifold \( X_6 \) in which the solution exists. The method of constructing the solution is based on a new property of the Pfaffians.

Solving the Vlasov equation with an arbitrary plasma-confining electromagnetic force field is a rather complicated problem. The numerical approaches aiming at constructive solutions for design purposes of controlled thermonuclear fusion machines have sometimes to use various simplifications. For these reasons it appears still nowadays interesting to dispose of analytical procedures to construct exact solutions of the Vlasov equation.

In the present note a solution is given for an arbitrary electromagnetic force field configuration given in advance. The collision term has been completely omitted.

For notational convenience the seven independent variables \((x, y, z, v_1, v_2, v_3, t)\) will be represented by the vector \( x := (x_i)_{i = 1, \ldots, 7}, \ x \in \Omega \) and \( \Omega \) is the subset of \( R^7 \) in which the distribution function does not vanish.

In addition the definitions will be used: \((u_1, u_2, u_3) := (v_1, v_2, v_3)\) for the velocity components and \((u_4, u_5, u_6) := (a_1, a_2, a_3)\) for the acceleration components \(u_7 = 1\).

It is also advantageous as long as we are dealing with toroidal devices to introduce the equation of a torus surface \( P = 0 \) with

\[
4R^2(x_1^2 + x_2^2) - (x_1^2 + x_2^2 + x_3^2 + R^2 - R'^2)^2 = : P,
\]

where \( R, R' \) are the major and the minor radii of the torus. Moreover, it is a feature of the physical situation that both the poloidal and the toroidal magnetic fields are in

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most cases considered as limited in the interior of a torus. For this reason it is expedient
to define the characteristic function

\[
\theta(P) = \begin{cases} 
1, & x \in T, \\
0, & \text{otherwise}.
\end{cases}
\]

\(T\) is the subset of \(R^3\) which is the interior of the torus, eq. (1).

It is assumed in the present note that \(u_i(x) \neq 0\) for \(i = 4, 5, 6\), so that possible sin-
gular surfaces of the differential operator, \(\mathcal{L}'\), are excluded

\[
\mathcal{L}' := \sum_{i=1}^{7} u_i(x) \partial_i.
\]

Moreover, the distribution function satisfies the boundary conditions:

\[
f(x_1, \ldots, x_{j-1}, x_j^0, x_{j+1}, \ldots, x_7) = f'(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_7),
\]

where \(x_j^0\) is on the boundary \(\partial \Omega_j\) of \(\Omega\) and \(f'\) is independent of \(x_j\). The function on the
l.h.s. of eq. (3) is denoted by \(f_j\). More generally, \(f_1, \ldots, f_7\) will denote the function \(f\) in which
the arguments \(x_1, \ldots, x_m\) have been replaced by their respective values on the boundaries
\(\partial \Omega_j, \cup \Omega_i = \Omega_j f_j\) is constant on \(\partial \Omega_j\). If the distribution function, \(f(x)\), obeys the
equation

\[
\mathcal{L}f = 0,
\]

then

\[
f(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = f_1(x_1, x_2, x_3, x_4, x_5, x_6),
\]

where \(f_1\) is an arbitrary function of all independent variables, but the time variable
and \(x_7\) is a constant time value.

Having fixed the above notation one can write the solution of eq. (3) as

\[
f(x) = \frac{1}{6} \sum_{i=1}^{7} f_i(x_1, \ldots, x_7) + C, \quad C := f_1^{34567},
\]

with the condition

\[
u(x) = 0,
\]

and

\[
u(x) := \sum_{j=1}^{7} \sum_{i \neq j} u_i(x) \partial_j \sum_{j=1}^{7} f_i,
\]

eq (7) defines a manifold, \(X_{6, 1}\), in which the distribution function is embedded.

Using the properties of the generalized function \(\delta(u(x))\), one can write eqs. (6), (7)