Periodic Solutions from Quaternionic Bifurcation (*).

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In dealing with bifurcation problems in the presence of symmetry (1,2), when the group representation involved in the bifurcation equation is irreducible (in real sense), it is known that the extended Schur lemma (4) leads to only three structurally different types of bifurcations:

i) the representation $T$ is irreducible also in the complex sense, and, therefore, the linearized part of the bifurcation equation is given simply by a multiple of the identity operator;

ii) the set $\mathcal{U}(T)$ of the operators which commute with the representation is generated by two independent operators: this possibility corresponds to the Hopf bifurcation problem, and in this case there is an underlying symmetry described by the group $S0_2$, fully examined in ref. (3);

iii) finally, the involved representation $T$ is of quaternionic type, and the set $\mathcal{U}(T)$ of the intertwining operators is generated by 4 independent operators.

We will consider here the third case, which appears to be less studied in the literature

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(4) A. KIRILLOV: Éléments de la théorie des représentations (MIR, Moscow, 1974).
(cf. (3, 4, 6)) and show that it leads in particular to the existence of bifurcating periodic solutions.

Let $u = u(t)$, with $u \in \mathbb{R}^4$, and consider the problem

$$
\dot{u} = f(\lambda, u) = L(\lambda)u + h(\lambda, u),
$$

where $\lambda$ is a real parameter, $f: \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^4$ a given analytical map, and $L(\lambda) = f'_u(\lambda, 0)$ its linearized part. As usual in bifurcation problems, it is assumed $f(\lambda, 0) = 0$, so that $u = 0$ is the "trivial" solution of (1). Suppose now that (1) is covariant with respect to the real 4-dimensional irreducible representation of the group $SU_2$, with Lie generators represented by

$$
G_1 = \frac{1}{2}
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{bmatrix},
G_2 = \frac{1}{2}
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{bmatrix},
G_3 = \frac{1}{2}
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}.
$$

This representation $T$ is precisely of quaternionic type: in fact the following operators:

$$
K_3 = I =
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
K_1 =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix},
K_2 =
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{bmatrix},
K_3 =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0
\end{bmatrix},
$$

commute with $T(\lambda, \sigma)$. The assumption about the covariance of (1)

$$
f(\lambda, T\sigma) = Tf(\lambda, u), \quad T = T(\sigma),
\quad \sigma \in SU_2,
$$