The Weak Semi-Differentiability in Quantum Mechanics.

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Summary. – The concept of weak semi-differentiability is strengthened by removing a defect from an earlier definition by Fonte and a serious weakness in the strengthened concept is demonstrated by a counter example.

Fonte (1) by developing further an idea originally due to Sharma and Rebelo (2) defined the concept of weak semi-differentiability on a complex Banach space. The purpose of this paper is to clarify and strengthen this concept in the new calculus on complex Banach spaces being developed by Fonte and co-workers (3,4) in Italy and Sharma and co-workers (5-7) in England.

We first point out that $E(T)$ of Fonte (see (1) p. 201) does not satisfy the requirement of being defined on an open set of a Banach space and hence does not satisfy one of the criteria of semi-differentiability. This $E(T)$ is defined to be $\langle \Psi | H | \Psi \rangle$ where $H$ is a semi-bounded self-adjoint operator on a Hilbert space $X$. Since $H$ is semi-bounded, it is unbounded and therefore cannot be defined on the whole of the Hilbert space. If $H$ were defined on an open set, by linearity the domain $(H)$ of $H$ will be the whole of the Hilbert space which is impossible.

The following definition of weak semi-differentiability overcomes these objections and in this definition $E(T)$ is weakly semi-differentiable.

Definition (weak semi-differentiability). Let $D(f)$ be a subset of a complex Banach space $X$ such that $D(f) = M \cap U$, where $M$ is a dense vector subspace of $X$ and $U$ is an open subset of $X$. A function $f: D(f) \rightarrow Y$, where $Y$ is another Banach space, is said to be weakly semi-differentiable at a point $u \in D(f)$, if there exists a bounded additive

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map $f_u^{(aw)}$ from $M$ to $Y$ such that for any $h \in M$

$$\lim_{t \to 0} \frac{f(u + th) - f(u)}{t} = f_n^{(aw)}(h),$$

the map $f_u^{(aw)}$ if it exists is called the weak semi-derivative of $f$ at $u$.

**Remarks.** 1) Let $u \in D(f)$. Since $D(f) = M \cap U$ for each $h \in M$, there exists a real number $\delta_h > 0$, such that $u + th \in D(f)$ for $|t| < \delta_h$.

2) $M$ being dense in $X$, the map $f_u^{(aw)}$ can be uniquely extended by continuity to the whole of $X$: $f_u^{(aw)}$ can be, without loss of generality, identified with this extension.

3) When $X$ is a Hilbert space and $Y = C$, by the Riesz representation theorem (see (2)) we can write

$$f_u^{(aw)} = \langle \cdot | g_1 \rangle + \langle g_2 | \cdot \rangle,$$

where $g_1$ and $g_2$ are fixed vectors in $X$.

4) Our definition does not require $f$ to be continuous, but for a fixed element $u \in D(f)$ and a fixed $h \in M$, the map

$$g: ]- \delta_h, \delta_h[ \to Y$$

defined by

$$g(t) = f(u + th)$$

is continuous at $t = 0$, if $f_u^{(aw)}$ exists.

5) $f$ need to be defined on an open set.

One major disadvantage of working with weak semi-differentiability is that for the stationary points of $f$, that is points $u \in X$ satisfying

$$f_u^{(aw)} = 0,$$

the following conditions on the second weak semi-derivative $f_u^{(2aw)}$:

$$f_u^{(2aw)}(h, h) \geq \mu \|h\|^2$$

for some $\mu > 0$ does not guarantee that $f$ achieves a minimum at $u$, as can be easily seen from the following counter example.

Let $(e_i)$ be an orthonormal basis in a Hilbert space $X$. Define a symmetric map $B$ by

$$Be_i = ie_i.$$ 

Extend $B$ by linearity to all finite linear combinations of the $e_i$'s. Then $D(B)$ is a vector subspace dense in $X$. Now define

$$f: D(B) \to C$$