A Trick for the Computation of Some Non-Gaussian Path Integrals.

S. Cecotti
Scuola Normale Superiore - Pisa
Istituto Nazionale di Fisica Nucleare - Sezione di Pisa

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Summary. -- We introduce a technique for the exact computation of any functional integral corresponding to a (Euclidean) Lagrangian of the form

\[ \mathcal{L} = \frac{1}{2} \dot{\varphi}^2 + \frac{1}{4} \mu^2 \exp[2 \lambda \varphi] + \mu \exp[\lambda \varphi] + \sum_{i=1}^{N} \psi_i \left[ \dot{\psi}_i + \eta_i \exp[\lambda \varphi] \psi_i + m_i \psi_i \right]. \]

The trick may be of use for other classes of path integrals, provided that their supersymmetric extensions have a local Nicolai mapping.

One of the main technical problems in theoretical physics is to compute functional integrals. It seems that only Gaussian integrals can be computed exactly. In this paper we show that there is at least another class of path integrals which can easily be computed in a closed form. These are the Liouville integrals in one dimension

(1) \[ Z = \int [d\varphi \, d\psi \, d\bar{\psi}] \exp \left[ -\int_{0}^{\beta} \mathcal{L} \, d\tau \right], \]

where the (Euclidean) Lagrangian \( \mathcal{L} \) is

(2) \[ \mathcal{L} = \frac{1}{2} \dot{\varphi}^2 + \frac{1}{4} \mu^2 \exp[2 \lambda \varphi] + \mu \exp[\lambda \varphi] + \sum_{i=1}^{N} \psi_i \left( \dot{\psi}_i + m_i \psi_i + \eta_i \exp[\lambda \varphi] \psi_i \right). \]

(\( \varphi \) is a bosonic field and \( \psi_i \) are fermionic), with periodic boundary conditions for the field \( \varphi \) and with generalized periodic boundary conditions for the fermions

(3) \[ \psi_i(\beta) = \exp[iQ_i] \psi_i(0). \]
As is well known, if $Q_j = \pi$ for all $j$, $Z$ equals the partition function

\begin{equation}
\text{Tr} \exp \left[ -\beta H \right].
\end{equation}

However, we stress that our trick is more general and it may well be of use for other cases in which a stochastic representation for the integral exists, such as the Polyakov string \(^{(1)}\).

Consider the Lagrangian

\begin{equation}
\mathcal{L} = \frac{1}{2} \dot{\psi}^2 + \frac{1}{2} \left[ V(\psi) \right]^2 = \frac{1}{2} \left[ \dot{\psi} - V(\psi) \right]^2 + \text{total derivative}
\end{equation}

from the theory of the local Nicolai mapping \(^{(2)}\) we know that

\begin{equation}
\int [d\psi] \exp \left[ -\beta \mathcal{L} \right] = \int [d\psi] \delta(h - \psi - V(\psi)) \exp \left[ -\beta \int_0^1 \dot{h}^2 \right]
\end{equation}

as the reader can check by doing the integral over $h$ on the r.h.s. of eq. (6). Then

\begin{equation}
\text{l.h.s. of eq. (6)} = \int [d\dot{h}] \exp \left[ -\frac{1}{\beta} \int_0^1 \dot{h}^2 \right] \left| \text{det} \left[ d_t + V'(\psi) \right] \right|^{-1} \left| \dot{\psi} + V(\psi) \right| = \frac{1}{\Sigma} \int [d\dot{h}] \exp \left[ -\frac{1}{\beta} \int_0^1 \dot{h}^2 \right] \left| \text{sinh} \left[ \frac{1}{\beta} \int_0^1 V'(\psi) \right] \right|^{-1} \left| \dot{\psi} + V(\psi) \right|
\end{equation}

where $\Sigma$ is the image of the map

\[ \dot{\psi} + V(\psi) = h(\tau) \]

in the functional space. In eq. (7) we used standard Floquet index arguments.

Equations (6) and (7) illustrate our main trick, which we call the stochastic form of the path integral. The theory of the Nicolai mapping \(^{(2, 3)}\) in supersymmetric theories gives the conditions under which a given bosonic theory (and its fermionic extensions) has a stochastic form for the integral. There is a special case, the Liouville model in two dimensions which has a stochastic representation, even if its supersymmetric counterpart has no local mapping, due to its classical equivalence with the massless free theory.

If ref. \(^{(4)}\) it was shown that the map

\begin{equation}
\dot{\psi} + \mu \exp \left[ \lambda \psi \right] + l = h
\end{equation}

was surjective, and (if $\mu > 0$) the image was the space of all periodic functions $h(t)$

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