On the Quantization of Classical Systems.

J.-P. Amiet

Institut für Theoretische Kernphysik der Universität - Bonn

(ricevuto il 17 Dicembre 1962)

Summary. — It is proved that no quantization rule can be invariant under all canonical transformations of the classical variables p and q. The maximum invariance of a quantization rule is determined.

1. — Introduction.

While the mathematical frameworks of classical and quantum mechanics have both solid and well-understood basis, the fundamental connections between the two theories have not yet met the same point of achievement.

The more widely known connections, namely the canonical quantization and the theorem of Ehrenfest, have undoubtedly a provisional character. That the canonical quantization yields so good results rises more astonishment than conviction, considering the fact that it is of no help in quantizing the simplest dynamical system expressed with arbitrary canonical variables. As regards the theorem of Ehrenfest, it is not evident that the result of its application is independent from the frame of reference.

These examples indicate already that the essential property of any fundamental relation connecting classical and quantum mechanics must be its independence with respect to the choice of the coordinate system. This fact introduces in a natural way the group-theoretical approach to the question, since the passage from any system of canonical co-ordinates to another is obtained by the help of a transformation belonging, in classical mechanics, to the «group» $\mathcal{G}$ of canonical transformations (*), and in quantum mechanics

(*) We call $\mathcal{G}$ a «group», though it is not a true group but only a local group of local transformations.
to the group $\mathcal{G}$ of unitary transformations. An essential advantage of this approach is that in both theories the time development of a dynamical system can be expressed by means of a family of transformations which belong to the corresponding group of invariance.

We shall discuss, in this paper, the possibility of extending the canonical quantization; on using the fact that $\mathcal{C}$ and $\mathcal{G}$ are Lie groups (in a generalized sense), we shall prove rigorously that there exists no fully invariant quantization rule, thus no fundamental extension of the canonical quantization.

None the less, a quantization rule which has a restricted invariance and applies to a given class of hamiltonians will still be of great importance. This research has been made in several papers devoted to quantization (1). We shall content ourselves, here, in determining the largest invariance any quantization rule can have.

In Section 2 a summary is given of the properties of canonical and unitary transformations that will be needed. In Section 3 the invariance of quantization is extensively discussed; the canonical transformations are generally defined not everywhere, a fact which could be an argument for the reduction of the required invariance. In Section 4 the above-mentioned result is proved.

2. – Mathematical structures of classical and quantum mechanics.

2.1. **Summary of some mathematical features of classical mechanics.** – In this section we assemble some definitions and properties concerning the classical canonical transformations (C.T.), which will be necessary for our purpose. Most of them are extensively treated in two papers (2,3), to which we refer the reader (*).

2.1.1. **Definition of canonical transformations.** Consider a dynamical system with $n$ degrees of freedom. Its behaviour is described in hamiltonian mechanics by $2n$ functions of the time,

$$ p_k(t), \quad q_k(t), \quad (k = 1, \ldots, n), $$


(2) K. Bleuler: to be published.


(*) In particular for the proofs of the theorems.