Local Duality and the Thomas-Fermi Model (*).  

H. SELLMANN  

Department of Theoretical Physics, Royal Institute of Technology  
S-100 44 Stockholm, Sweden  

(ricevuto il 18 Aprile 1984)  

PACS. 12.90. -- Miscellaneous theoretical ideas and models.  

Summary. -- The Thomas-Fermi model applied to the sum over resonances of quark-antiquark pairs produced in $e^+e^-$ annihilation is shown to directly yield the parton model result for the cross-section.

In $e^+e^-$ annihilation into hadrons the cross-section can be calculated either as a sum over the narrow vector meson resonance states or as the production of a «quasi»-free quark-antiquark pair. This latter is often called the parton model result.

However, when all the resonances are summed over, the cross-section so calculated and suitably smeared over in energy is expected to reproduce the parton model calculation. This property is called local duality (1). It has been investigated in potential models by several authors (2-4). The cross-section is proportional to the widths of the resonances and hence to the wave functions at the spatial origin. By making use of Thomas-Fermi model for quark-antiquark bound state, BELL and PASUPATHY (2) obtained a general expression for the first nonvanishing derivative of the wave function at the origin in the nonrelativistic limit. For $S$-waves this formula leads to the parton model result for the cross-section in the limit of close level spacing.

The purpose of the present note is to point out that, by using the Thomas-Fermi model for the sum of the excited states of the quark-antiquark pair, one immediately gets the parton model result. This procedure can be extended to the relativistic realm, where the sum of the $S$- and $D$-waves satisfies the local duality property to all orders.

(*) Work supported by the Swedish Natural Science Research Council (NFR), contract No. F-UR3281-104.  
in the velocity. This extends an earlier result of Tainov (4) who showed local duality to $O(v^2)$ in the velocity.

The $e^+e^-$ annihilation cross-section is

$$\sigma(E) = 6\pi^2 \sum_j M_j^2 \Gamma_j \delta(E - M_j)$$

in the narrow resonance approximation. Here $M_j$ is the mass of the $j$-th resonance and

$$\Gamma_j = 16\pi^2 e_q^2 |\Psi_j(0)|^2 M_j^{-1}.$$  

The wave function $\Psi_j(\vec{x})$ satisfies the Schrödinger equation

$$H \Psi_j(\vec{x}) = (2m + p^2/m + V(\vec{x})) \Psi_j(\vec{x}) = E_j \Psi_j(\vec{x}),$$

where $m$ is the quark mass and $V(\vec{x})$ the confining interquark potential.

The cross-section can now be written in the following form:

$$\sigma(E) = 96\pi^3 x^2 e_q^2 E^{-4} \frac{d\rho_E(0)}{dE},$$

where

$$\rho_E(\vec{x}) = \sum_j \theta(E - M_j) |\Psi_j(\vec{x})|^2 = \langle \vec{x} | \theta(E - H) | \vec{x} \rangle$$

is the state density at energy $E$. By putting $n_F = n(E_F)$, where $E_F$ is the Fermi energy, we regain the more conventional form of the density

$$\rho_{E_F}(\vec{x}) = \sum_{j=0}^{n_F} |\Psi_j(\vec{x})|^2.$$  

The semi-classical approximation of $\rho_{E_F}(\vec{x})$ is given by

$$\rho_{E_F}(\vec{x}) = \int d^3p (2\pi)^{-3} \theta(p_{E_F}(\vec{x}) - p) = p_{E_F}(\vec{x})/\delta \tau^2,$$

where $p_{E_F}(\vec{x}) = \sqrt{m(E - 2m - V(\vec{x}))}$.

When $\vec{x} = 0$, the sum in (5) is only over the $S$-states. Hence, when $V(0) = 0$, we immediately get

$$\sigma(E) = \frac{4\pi^2/3 E^2}{3 e_q^2 3\nu(0)/2}.$$  

This agrees with the (nonrelativistic) parton model result when we identify $\nu(0) = p(0)/m$ with $v$.

We next observe that the same treatment can be applied to the quark-antiquark system when it is described by a relativistic local potential equation replacing the nonrelativistic Schrödinger equation:

$$H \Psi_j(\vec{x}) = (2E(\vec{p}) + V(\vec{x})) \Psi_j(\vec{x}) = M_j \Psi_j(\vec{x}),$$

where $E(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$. 