On Wigner’s Little Groups and Scale Transformations.

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Summary. Little group structures are discussed when Poincaré transformations are extended by dilatations. Only lightlike four-momenta do enter into this study.

When scale transformations (or dilatations) are studied together with Poincaré (or inhomogeneous restricted Lorentz) transformations, it is well known that we get an eleven-parameter Lie group called the Weyl group or the similitude group in \((3+1)\) space \((SIM_{3,1})\). Its Lie algebra is generated by the infinitesimal operators \((\{P_\mu\}, \{M_\mu\} = (J, K)\) and \(D\)) associated with space-time translations \((\mu = 0, 1, 2, 3)\), restricted Lorentz transformations (rotations \(J\) and boosts \(K\)) and dilatations, respectively, the corresponding parameters being denoted by \(\alpha, \omega^{\mu\nu}\) and \(\varrho\). The nonzero commutation relations are well known and can be summarized in the following form:

\[
\begin{align*}
[M_\mu, M_\nu] &= i(g_{\mu\sigma} M_\rho + g^{\rho\sigma} M_\mu - g^{\rho\nu} M_\sigma - g_{\nu\sigma} M_\mu), \\
[M_\mu, P_\nu] &= i(g^{\rho\nu} P_\mu - g^{\rho\mu} P_\nu), \\
[P_\mu, D] &= iP_\mu, \\
[M_\mu, D] &= 0, \\
[P_\mu, P_\nu] &= 0.
\end{align*}
\]

They evidently form a subalgebra of the conformal algebra studied in particular by Mack and Salam (1). In this note, we refer to Minkowski space-time characterized by the metric tensor

\[
g_M = (g_{\mu\nu}) = \text{diag} (1, -1, -1, -1).
\]
The action of the Weyl group on contravariant space-time events reads

\[ x'^\mu = x^\mu + g x^\mu + \omega^\nu \epsilon x^\nu + a^\mu, \]

although in covariant co-ordinates we have

\[ x'_\mu = x_\mu - q x_\mu + \omega^\nu_\mu x_\nu + a_\mu, \]

due to the fact that \( g_\mu \) is not scale invariant:

\[ g'^{\mu\nu} = (1 + 2q)g^{\mu\nu} \quad \text{and} \quad g'_{\mu\nu} = (1 - 2q)g_{\mu\nu} \]

in this context, while it is when only Poincaré transformations are considered.

Owing to these generalities, let us study here Wigner's little groups (\(^7\)) when Poincaré transformations are extended by dilatations. First, we notice in correspondence with (3) and (4) that the four-momenta \( p \) do transform according to

\[ p'^\mu = p^\mu + q p^\mu + \omega^\nu p^\nu, \]

or

\[ p'_\mu = p_\mu - q p_\mu + \omega^\nu p_\nu, \]

so that we have to make a choice at the level of covariant or contravariant components of \( p \) in order to study the corresponding little groups. But if space-time events \( x = (x^\mu) \) are under study through (3), we know that the four-momentum is essentially associated with covariant derivatives \( \partial_\mu = \partial_\mu x^\mu \), so that the meaningful transformation law on \( p \) is (7). In the same way, eqs. (4) and (6) are physically relevant.

Then let us consider the Weyl transformations leaving such a four-momentum \( p \) invariant, i.e. the extended Wigner little groups of \( p \). With eqs. (7) we get the invariance conditions

\[ \omega_\mu p_\nu = q p_\mu, \]

implying that

i) \( p^\mu p_\mu = 0 \), which means we only deal with lightlike four-vectors associated with zero-mass particles (we do not consider the continuous case);

ii) there are only four independent parameters characterizing the extended little group (only three of the relations (8) are linearly independent);

iii) the four generators are given by

\[ G = P \wedge J + P^0 K - P D, \quad G_4 = P \cdot J; \]

iv) their algebra is given by

\[ [G^i, G^j] = -i \epsilon^{ijk} (P \wedge G)^k; \quad [G_4, G] = -i P \wedge G. \]