Conformal Point Transformations, Invariance Conditions and Lie Derivatives.

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Summary. Conformal co-ordinate and point transformations are considered in order to express invariance conditions on different tensor fields and, more particularly, on metric tensors. Lie derivatives of these fields are re-examined in the context of conformal point transformations leading to conformal Killing equations and to invariance conditions on dynamical fields in gauge theories.

Modern differential geometry (1) enters more and more currently into the formulation of particle physics and, more particularly, into gauge theories. When invariances, symmetries and conservation laws are under study, the concept of Lie derivatives (1) plays a fundamental role as developed for example by Forgacs and Manton (2) and Jackiw (3,4). In the discussion of tensor (5), spinor (6) or gauge (7) fields invariant under different symmetry groups, these Lie derivatives are also of fundamental interest. In fact, let us recall that, if the symmetry group is a Lie group \( G \), the requirement of invariance is expressed by the condition that the Lie derivatives of \( p \)-forms (for example) with respect to the vector fields induced by the one-parameter subgroups of \( G \) should vanish. Applications of this kind when (space-time) conformal co-ordinate transformations are considered can be found elsewhere (5): in fact, symmetric tensors of order 2, 2-forms or skew symmetric tensors of order 2, 1-forms or covariant four-vectors and scalar densities have been determined when invariance is required under the conformal group of space-time or under some of its subgroups. Moreover, parallel develop-

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ments (8,9) have also been done in the context of conformal point transformation (10) discussed by Fulton, Rohrlich and Witten.

Here we just want to relate the invariances under co-ordinate and Weyl transformations discussed by Jackiw and Manton (2) with the invariances under point transformations when the conformal context is taken into account. Through conformal Killing equations on invariant metric tensors, we get simple relations of particular interest with a view to study invariances of dynamical fields in gauge theories.

Conformal co-ordinate transformations can be characterized through real infinitesimal parameters \( a, \omega, \varrho \) and \( c \) associated with space-time translations, restricted homogeneous Lorentz transformations, dilatations and special conformal transformations, respectively. Then, if one refers to Minkowski space characterized by the (constant) metric tensor \( G_{\mu\nu} = \{ \eta_{\mu\nu} \} = \text{diag} \{ 1, -1, -1, -1 \} \), the infinitesimal conformal co-ordinate transformations take the form

\[
x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu \quad (\mu = 0, 1, 2, 3)
\]

with

\[
\xi^\mu = a^\mu + \omega^\mu x^0 + \varrho x^\mu + 2x^\mu (c \cdot x) - c^\mu x^2.
\]

Under these transformations, the invariance conditions imposed on an arbitrary symmetric covariant tensor \( S(x) = \{ S_{\mu\nu}(x) \} \) of order 2 correspond to

\[
L_X S(x) = 0, \quad X = \xi^\mu \partial_\mu,
\]

i.e. to the annulation of the Lie derivative of \( S \) with respect to the vector fields \( X \). Explicitly, we have

\[
L_X S_{\mu\nu}(x) = \xi^\rho \partial_\rho S_{\mu\nu}(x) + (\partial_\mu \xi^\rho) S_{\nu\rho}(x) + (\partial_\nu \xi^\rho) S_{\mu\rho}(x) = 0.
\]

When eqs. (1) and (2) are taken into account, we get (5)

\[
\mathcal{D} S(x) - \omega \times S(x) - 2(\varrho \times x) \times S(x) - \left[ \varrho + 2(c \cdot x) \right] [2 + x \cdot \nabla] S(x) + x^\mu (c \cdot \nabla) S(x) = 0,
\]

where

\[
\mathcal{D} \equiv x \cdot \omega \cdot \nabla - a \cdot \nabla \quad (x \cdot \omega \cdot \nabla = - \omega \cdot x \cdot \nabla),
\]

\[
(\omega \times S)_{\mu\nu} = \omega_\mu S_{\nu\sigma} + \omega_\nu S_{\sigma\mu},
\]

\[
(x \cdot c) \sigma_\sigma = x^\sigma c_\sigma - c_\sigma \sigma,
\]

\[
[(x \times c) \times S]_{\mu\nu} = (x \times c)_\mu S_{\nu\sigma} + (x \times c)_\nu S_{\mu\sigma}.
\]

Let us notice that, if \( S \) is a metric tensor, the invariance condition (4) can be written as

\[
\xi_{\mu\nu} + \xi_{\nu\mu} = 0,
\]