A Systematic Search for Nonrelativistic Systems with Dynamical Symmetries.

PART I: The Integrals of Motion.

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Summary. — The purpose of this investigation is to give a general discussion of so-called accidental degeneracy and the corresponding dynamical invariance groups in quantum mechanics. In the present paper the existence of pairs of commuting integrals of motion not higher than second order in the derivatives is considered and related to the separation of variables in the Schrödinger equation. All Hamiltonians are found which allow three or four integrals of motion, two of which are related to the angular momentum and its projection.

1. - Introduction.

The importance of various types of symmetries in classical and quantum physics is generally acknowledged and this subject has been considered with renewed interest in the past few years (partly in connection with the symmetry approach to elementary-particle physics).

The original symmetries used in physics were geometrical symmetries connected with the properties of the usual space-time continuum. We shall be interested in a different type of symmetry—in so-called dynamic invariance groups, connected with specific types of interactions. Contrary to geometrical

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symmetries, which basically reflect the fact that physical processes do not depend on the place in space and time where they occur, dynamical symmetries just imply that certain physical laws, i.e. Lagrangians, matrix elements etc., are invariant with respect to the transformations of the corresponding group (see, e.g. (1)).

In this sense we call the well-known $O_4$ symmetry for the bound states of the hydrogen atom (2) or $SU_3$ symmetry of the harmonic oscillator (3) dynamical invariance groups. In this article we shall not consider spectrum-generating groups, i.e. dynamical noninvariance groups containing different energy levels in a single irreducible representation (4).

This paper is part of a systematic investigation, the aim of which is to find, with certain restrictions, all potentials allowing a dynamical (or higher) invariance group. Our approach will be in the framework of nonrelativistic quantum mechanics. Quantum mechanics is more convenient than classical physics for group-theoretical considerations, because the Hilbert space of wave functions is a natural space for realizing the representations of the corresponding symmetry groups. Indeed, in quantum mechanics the existence of a dynamical invariance group demonstrates itself by the «accidental» degeneracy of the energy levels of the system. This degeneracy is then described by the irreducible representations of the dynamical invariance group, the generators of which all commute with the Hamiltonian.

In classical mechanics the meaning of dynamical invariance groups is less straightforward. In a system with $s$ degrees of freedom, $2s-1$ integrals of motion always exist. Combining these integrals of motion and forming arbitrary functions of the integrals, closed algebras (with respect to the Poisson bracket operation) can be constructed for very general potentials (5) so that the existence of an algebra of integrals of motion has in itself little physical meaning. In our opinion, the existence of dynamical symmetries in classical mechanics is connected with the existence of closed paths (it was proved as early as 1873 by Bertrand (6) that the only centrally symmetrical potentials in which all finite paths are closed are the Coulomb potential and

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(4) Y. Dothan, M. Gell-Mann and Y. Ne'eman: Phys. Lett., 17, 148 (1965);