THE MAXIMAL VARIATION OF A BOUNDED MARTINGALE

BY

JEAN-FRANCOIS MERTENS AND SHMUEL ZAMIR

ABSTRACT

Let \( \chi^n = \{X_i\}^n_0 \) be a martingale such that \( 0 \leq X_i \leq 1 \); \( i = 0, \ldots, n \). For \( 0 \leq p \leq 1 \) denote by \( \mathcal{M}_p^n \) the set of all such martingales satisfying also \( E(X_0) = p \). The variation of a martingale \( \chi^n \) is denoted by \( V(\chi^n) \) and defined by \( V(\chi^n) = E(\sum^n_{i=0} |X_{i+1} - X_i|) \). It is proved that

\[
\lim_{n \to \infty} \left( \sup_{\chi^n \in \mathcal{M}_p^n} \left[ \frac{1}{\sqrt{n}} V(\chi^n) \right] \right) = \phi(p),
\]

where \( \phi(p) \) is the well known normal density evaluated at its \( p \)-quantile, i.e.

\[
\phi(p) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \quad \text{where} \quad \int_{-\infty}^{x_p} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx = p.
\]

A sequence of martingales \( \chi^n_0, n = 1, 2, \cdots \) is constructed so as to satisfy \( \lim_{n \to \infty} (1/\sqrt{n}) V(\chi^n_0) = \phi(p) \).

1. Introduction

For a martingale \( \chi^n_0 = \{X_i\}^n_0 \) we define the variation by \( V(\chi^n_0) = E(\sum^n_{i=0} |X_{i+1} - X_i|) \). We are interested in this variation for bounded martingales, say \( 0 \leq X_i \leq 1 \), \( i = 0, 1, 2, \cdots \). For any \( p: 0 \leq p \leq 1 \) denote by \( \mathcal{M}_p^n \) the set of all \( n \)-martingales bounded in \([0,1]\) and satisfying \( E(X_0) = p \) (\( E(X) \) denotes the expectation of \( X \)).

A rather easy consequence of a well known property of martingales and the Cauchy–Schwartz inequality is that

\[
V(\chi^n_0) \leq \sqrt{p(1-p)} \cdot \sqrt{n}
\]

for every \( \chi^n_0 \in \mathcal{M}_p^n \). In particular if \( \{X_i\}^\infty_0 \) is an infinite martingale with \( E(X_0) = p \) and \( \chi^n_0 \) is its truncation at stage \( n \), then (1.1) holds for \( n = 1, 2, \cdots \). However this
is not the strongest statement possible in this case since from the convergence of \( \{X_n\}_{n=0}^\infty \) it can be shown that for any such \( \infty \)-martingale

\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} V(X_0) = 0.
\]

The question we are interested in is: Is \( \sqrt{n} \) the least upper bound for the order of magnitude of \( V(X_0) \)? Since obviously there are \( n \)-martingales with \( V(X_0) \) of lower order of magnitude, the question is: Is there a function \( f(p) \); \( f(p) > 0 \) for \( 0 < p < 1 \); such that for each \( 0 \leq p \leq 1 \) and a positive integer \( n \) there exists an \( n \)-martingale \( X_0^n \in \mathcal{M}^n_p \) satisfying

\[
V(X_0^n) \geq f(p) \sqrt{n} ?
\]

Notice that in view of (1.2) it is impossible to satisfy (1.3) with the \( X_0^n \) being the truncations of the same \( \infty \)-martingale. An affirmative answer to the above stated question would imply: There exists \( f(p) \); \( f(0) = f(1) = 0 \) and \( f(p) > 0 \) for \( 0 < p < 1 \) such that

\[
f(p) \leq \sup_{X_0^n \in \mathcal{M}^n_p} \left[ \frac{1}{\sqrt{n}} V(X_0^n) \right] \leq \sqrt{p(1-p)}.
\]

It turns out that a result much stronger than (1.4) can be achieved, namely

\[
\lim_{n \to \infty} \sup_{X_0^n \in \mathcal{M}^n_p} \left[ \frac{1}{\sqrt{n}} V(X_0^n) \right] = \phi(p)
\]

where

\[
\phi(p) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2) ; \quad \int_{-\infty}^{\sqrt{p}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2) \, dx = p.
\]

Thus, not only is \( \sup_{X_0^n \in \mathcal{M}^n_p} [(1/\sqrt{n}) V(X_0^n)] \) bounded away from 0 but it is a converging sequence, the limit of which is, amazingly enough, the well known normal density function evaluated at its \( p \)-quantile.

A by-product of the proof of this result is a construction of a sequence of martingales \( X_0^n \in \mathcal{M}^n_p ; \ n = 1, 2, \ldots \) for which

\[
\lim_{n \to \infty} \left[ \frac{1}{\sqrt{n}} V(X_0^n) \right] = \phi(p).
\]

Our interest in the variation of bounded martingales came up through game theory. It turns out that the speed of convergence of the values of certain