EXISTENTIALLY CLOSED STRUCTURES
AND JENSEN'S PRINCIPLE ♦

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ABSTRACT
Assume \( V = L \) or even \( \mathcal{O}_\kappa \), there is no uncountable locally finite group which can be embedded in every uncountable universal locally finite group. Similar results hold for existentially closed groups and division rings.

0. Introduction

In [8] Shelah and I proved that in each cardinal \( \kappa > \aleph_0 \) there are \( 2^\kappa \) non-isomorphic universal locally finite groups. This answered a question of Kegel and Wehrfritz [7]. In fact we provided rather more detailed algebraic information. Namely, there is a locally finite group \( H \) of cardinal \( \aleph_1 \) such that in each cardinal \( \kappa > \aleph_0 \) there is a universal locally finite group \( G \) which has no subgroup isomorphic to \( H \). This gives two non-isomorphic universal locally finite groups of cardinal \( \kappa \), since it is known on general grounds that there is a universal locally finite group of power \( \kappa \) with \( H \) as a subgroup.

We found such an \( H \) which is 2-step solvable and of exponent 6. We wondered if \( H \) could be chosen abelian. Philip Hall [5] had shown that all countable locally finite groups are embeddable in all universal locally finite groups, so we posed the following

**Problem.** Which locally finite groups \( H \) are embeddable in all universal locally finite groups of cardinal \( \geq \text{card}(H) \)?

Let us call such groups *inevitable*. In the present paper I contribute to the above problem the following:

**Theorem.** Assume \( V = L \). There are no inevitable abelian groups of cardinal \( \aleph_1 \).

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So, assuming $V = L$, the 2-step solvable group mentioned above can be replaced by any locally finite abelian group of cardinal $N$.

I do not know if $V = L$ is needed for this.

The method applies not only to universal locally finite groups but also to existentially closed groups and division rings [6]. I give a uniform presentation for all three classes.

_Note added in proof, August 18, 1976._ Much stronger results have now been obtained for groups and locally finite groups. Independently of my work, Kenneth Hickin of Michigan State proved Theorem 1 for locally finite groups using only $ZFC$. Somewhat later, at the group theory meeting in Oxford, in July 1976, Shelah proved Theorem 1 for groups, using only $ZFC + CH$. Both workers use much "slower" (and more subtle) enumerations than I do, and in addition Shelah uses some very deep results of Ziegler on e.c. groups. So far there are no improvements in the case of division rings. The analogue of Shelah's result may need some version of Higman's embedding theorem for division rings.

Both workers obtained many other interesting results. For example, Shelah proved that there are no uncountable inevitable locally finite groups.

1. **Basic concepts**

I refer to [6] for all the model-theoretic ideas needed in this paper.

1.1. Let $\mathcal{C}$ be a class of structures for a countable language $\mathcal{L}$. $\mathcal{C}$ will be either

i) the class of existentially closed groups,

ii) the class of existentially closed division rings,

iii) the class of universal locally finite groups.

Note that (iii) is in fact the class of existentially closed locally finite groups, as pointed out in [8].

I will isolate certain axioms on the class $\mathcal{C}$, and use only these in the sequel.

The first is

(UC): $\mathcal{C}$ is closed under union of chains.

1.2. **Centralizers.** In each of the above cases we have the notion of the centralizer of a set of elements. I will look at this quite abstractly, as follows.

For each $\mathcal{M}$ in $\mathcal{C}$ we have a map

$$X \mapsto C_{\mathcal{M}}(X)$$

from the power set of $\mathcal{M}$ to itself.