THE COMBINATORIAL STRUCTURE
OF (m, n)-CONVEX SETS

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ABSTRACT
Let S be a closed subset of a Hausdorff linear topological space, S having no
isolated points, and let \( c_s(m) \) denote the largest integer \( n \) for which S is \((m,n)\)-
convex. If \( c_s(k) = 0 \) and \( c_s(k + 1) = 1 \), then

\[
c_s(m) = \sum_{i=1}^{k} \left( \frac{m + k - i}{k} \right)
\]

Moreover, if \( T \) is a minimal \( m \) subset of S, the combinatorial structure of \( T \) is
revealed.

1. Introduction
Throughout, the set \( S \) will be a subset of a Hausdorff linear topological space.
Employing the terminology used by Guay and Kay [2], for integers \( m, n \), we say
that \( S \) is \((m,n)\)-convex iff for each \( m \) distinct points of \( S \), at least \( n \) of the
\( \binom{m}{2} \) possible segments determined by these points are in \( S \). For convenience, when
\( 1 \geq m \geq 0 \), we say \( S \) is \((m,0)\)-convex. Thus the definition of \((m,n)\)-convex is
meaningful for any \( m \geq 0 \) and for \( \binom{m}{2} \geq n \geq 0 \). A set \( S \) is exactly \((m,n)\)-
convex iff \( S \) is \((m,n)\)-convex, and not \((m,n + 1)\)-convex, and \( c_s(m) \) will denote the unique
integer \( n \) for which \( S \) is exactly \((m,n)\)-convex.

For notational purposes, \( \sigma(k, m) \) will represent the following summation:

\[
\sigma(k, m) = \sum_{i=1}^{k} \left( \frac{m + k - i}{k} \right)
\]

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Finally, we will make use of the following familiar definitions:

For $x, y$ in $S$, we say $x$ sees $y$ via $S$ iff the corresponding segment $[x, y]$ lies in $S$. A subset $T$ of $S$ is visually independent via $S$ iff for every $x, y$ in $T$, $x \neq y$, $x$ does not see $y$ via $S$.

2. A formula for $c_4(m)$

For $S$ a closed $(p, q)$-convex set having no isolated points, $q \geq 1$, we are interested in the possible values which may be assumed by the sequence $(c_4(m): m \geq 2)$. Letting $k$ denote the largest integer for which $c_4(k) = 0$, the following theorems reveal that $c_4(m)$ is uniquely determined by $k$ for every $m$, and in fact $c_4(m) = \sigma(k, m)$.

**Theorem 1.** If $S$ is a closed $(m, n)$-convex set, $n \geq 1$, then $S$ is exactly $(m_0, 1)$-convex for some $m_0 \geq 2$.

**Proof.** Clearly $S$ has at most $j$ isolated points $z_1, z_2, \cdots, z_j$ where $j < m$. Letting $T = S \sim \{z_1, \cdots, z_j\}$, $T$ is $(m - j, n)$-convex. Let $m_0$ denote the smallest positive integer for which $c_4(m_0) > 0$. If $T$ is convex, the result is trivial, so without loss of generality assume $m_0 = 3$. We will show that $c_4(m_0) = 1$. Since $c_4(m_0 - 1) = 0$, there is a visually independent subset $\{x_1, \cdots, x_{m_0 - 1}\}$ of $T$ having $m_0 - 1$ members. Since $x_1$ is not an isolated point, there is an infinite net in $T \sim \{x_1\}$ converging to $x_1$. For some $y$ in this net, $[y, x_i] \subseteq T$ for every $i, 1 < i \leq m_0 - 1$. (Otherwise, there would be a subnet converging to $x_1$, each point of which sees via $S$ a particular $x_{i_0}$, and since $T$ is closed, $[x_1, x_{i_0}]$ would lie in $T$, a contradiction.)

Thus $\{x_1, \cdots, x_{m_0 - 1}, y\}$ is a set with $m_0$ members for which only one of the corresponding segments lies in $T$. We conclude that $c_4(m_0) = 1$ and $c_4(m_0 + j) = 1$.

**Remark.** It is interesting to note that if $S$ is not closed, the result fails. (See Example 1 of this paper.)

**Theorem 2.** Let $S$ be a closed set having no isolated points and with $c_4(k) = 0$, $c_4(k + 1) = 1$ for some integer $k$. Then

$$c_4(m) \leq \sum_{i=1}^{k} \left( \left\lfloor \frac{m + k - i}{2} \right\rfloor \right) = \sigma(k, m)$$

for every integer $m \geq 0$.

**Proof.** We exhibit an $m$ member subset of $S$ having at most $\sigma(k, m)$ corre-