FIXED POINT THEOREMS FOR NON-LIPSCHITZIAN MAPPINGS
OF ASYMPTOTICALLY NONEXPANSIVE TYPE†

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ABSTRACT

Let $X$ be a Banach space, $K$ a nonempty, bounded, closed and convex subset of $X$, and suppose $T : K \to K$ satisfies:

(*) for each $x \in K$, $\limsup_{i \to \infty} \{ \sup_{y \in K} \{ \| T^i x - T^i y \| - \| x - y \| \} \} \leq 0$.

If $T^N$ is continuous for some positive integer $N$, and if either (a) $X$ is uniformly convex, or (b) $K$ is compact, then $T$ has a fixed point in $K$. The former generalizes a theorem of Goebel and Kirk for asymptotically nonexpansive mappings. These are mappings $T : K \to K$ satisfying, for $i$ sufficiently large,

$\| T^i x - T^i y \| \leq k_i \| x - y \|$, $x, y \in K$, where $k_i \to 1$ as $i \to \infty$. The precise assumption in (a) is somewhat weaker than uniform convexity, requiring only that Goebel’s characteristic of convexity, $\varepsilon_0(X)$, be less than one.

Let $X$ be a Banach space, $K \subseteq X$. A mapping $T : K \to K$ is called asymptotically nonexpansive on $K$ [5] if there exists a sequence $\{k_i\}$ of constants such that $k_i \to 1$ as $i \to \infty$ and for which

$\| T^i x - T^i y \| \leq k_i \| x - y \|$, $x, y \in K$, $i \geq N_0$.

It was proved in [5] that if $X$ is uniformly convex and if $K$ is bounded, closed, and convex, then such a mapping must have a fixed point. This is, of course, a natural generalization of the fixed point theorem of Browder-Göhde-Kirk [1], [8], [11] for nonexpansive mapping.

Our purpose in this paper is twofold. First we substantially weaken the assumption of asymptotic nonexpansiveness of $T$ by replacing it with an as-

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assumption, (2) below, which may hold even if none of the iterates of $T$ is Lipschitzian. Although we assume that at least one of its iterates is continuous, the mapping itself need not be. In addition, we obtain one of our results in a class of spaces which properly includes the uniformly convex spaces.

Our second objective is to obtain an analogous result for compact convex $K$ with no underlying assumptions on the norm of the space. Again the assumption is that $T: K \to K$ satisfy (2) and $T^N$ be continuous for some $N$. This theorem provides a new result even for asymptotically nonexpansive mappings, and because $T$ is not assumed continuous it does not follow directly from the Schauder theorem.

A generalization of the result of [5] which retains the feature that iterates of $T$ are Lipschitzian but only requires that these Lipschitz constants be sufficiently near one (while perhaps being bounded away from one) is given in [6]. It is assumed that $X$ is uniformly convex in [6], but this result itself has subsequently been generalized in [7] to the wider class of spaces considered below.

The modulus of convexity of $X$ is the function $\delta: [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf \{1 - \frac{1}{2} \| x + y \| : x, y \in X, \| x \|, \| y \| \leq 1, \| x - y \| \geq \varepsilon\}.$$

Let

$$\varepsilon_0(X) = \sup \{\varepsilon : \delta(\varepsilon) = 0\}.$$

The number $\varepsilon_0(X)$ is called the characteristic of convexity of $X$ [4]. In Theorem 1 we assume $X$ satisfies $\varepsilon_0(X) < 1$. It is known (see Goebel [4]) that this implies $X$ is uniformly non-square, hence reflexive [10]. Also, $X$ is uniformly convex [2] if $\delta(\varepsilon) > 0$ whenever $\varepsilon > 0$; hence $\varepsilon_0(X) = 0$ for such spaces and so Theorem 1 holds for $X$ uniformly convex.

It is known (see [9], [12]) that the modulus of convexity is continuous and increasing on $[\varepsilon_0, 2)$ and moreover [13], [14], the inequalities

$$\| x \| \leq d, \| y \| \leq d, \| x - y \| \geq \varepsilon$$

imply

$$\frac{1}{2} \| x + y \| \leq (1 - \delta(\varepsilon/d))d.$$

For $x \in X$, $S(x; r)$ will denote the closed spherical ball $\{y \in X : \| x - y \| \leq r\}$.

In each of our theorems we assume $T: K \to K$ satisfies:

$$\text{for each } x \in K, \limsup_{i \to \infty} \sup_{y \in K} \{\| T^i x - T^i y \| - \| x - y \|\} \leq 0.$$