THEORIES OF LINEAR ORDER

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ABSTRACT
Let $T$ be a complete theory of linear order; the language of $T$ may contain a finite or a countable set of unary predicates. We prove the following results. (i) The number of nonisomorphic countable models of $T$ is either finite or $2^\omega$. (ii) If the language of $T$ is finite then the number of nonisomorphic countable models of $T$ is either 1 or $2^\omega$. (iii) If $S_1(T)$ is countable then so is $S_n(T)$ for every $n$. (iv) In case $S_1(T)$ is countable we find a relation between the Cantor Bendixson rank of $S_1(T)$ and the Cantor Bendixson rank of $S_n(T)$. (v) We define a class of models $\mathcal{S}$, and show that $S_1(T)$ is finite iff the models of $T$ belong to $\mathcal{S}$. We conclude that if $S_1(T)$ is finite then $T$ is finitely axiomatizable. (vi) We prove some theorems concerning the existence and the structure of saturated models.

Introduction

In this paper we deal with complete theories whose models are of the type $\mathcal{U} = \langle A, <^\mathcal{U}, P_1^\mathcal{U}, \ldots \rangle$ where $<^\mathcal{U}$ linearly orders $\mathcal{U}$, and $\{P_1, \ldots, P_n, \ldots\}$ is a finite or a countable set of unary predicates. In a well-known example Ehrenfeucht shows that, for every positive $n \neq 2$ there is a theory $T$ as mentioned above which has exactly $n$ nonisomorphic countable models. In Section 6 we shall show that every such $T$ has either finitely many nonisomorphic countable models or $2^\omega$ nonisomorphic countable models. We thus obtain a complete answer to the question: given a cardinal $\alpha$ is there a theory $T$, as mentioned above, such that $T$ has exactly $\alpha$ nonisomorphic countable models.

If the language of $T$ is finite we shall sharpen our result and prove that either $T$ is $\omega$-categorical or $T$ has $2^\omega$ nonisomorphic countable models.

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In Section 5 we characterize the complete theories of linear order $T$ whose language contains a fixed finite set of unary predicates, and for which $S_1(T)$ is finite. We define the class $\mathcal{S}'$, as the smallest class of models which contains all the models with a single element and which is closed under the following operations.

1. $s(\mathcal{A}, \mathcal{B}) = \mathcal{A} + \mathcal{B}$

2. $d(\mathcal{A}_1, \cdots, \mathcal{A}_n) = \sum_{r \in \mathbb{Q}} \mathcal{A}$, where $\mathbb{Q}$ is the ordered set of rationals and the family $\{\{r \mid \mathcal{A} \cong \mathcal{A}_i\} \mid i = 1, \cdots, n\}$ is a partition of $\mathbb{Q}$ consisting of dense subsets of $\mathbb{Q}$.

3. $z(\mathcal{A}) = \mathcal{A} \cdot \mathbb{Z}$, where $\mathbb{Z}$ is the ordered set of the integers.

We shall show: (i) For $T$, as above, the following conditions are equivalent.

Condition I. $S_1(T)$ is finite.

Condition II. $T$ has a model which belongs to $\mathcal{S}'$.

The following results will be then inferred. (ii) If $S_1(T)$ is finite then $T$ is finitely axiomatizable. (iii) For every $n$ the set $\{T \mid \| S_1(T) \| \leq n\}$ is finite. (i) and (ii) are related to [6] and [4].

Rosenstein in [6] showed that if we define $\mathcal{M}$ to be the subclass of $\mathcal{S}'$ which is closed only under operations (1) and (2) then $T$ is $\omega$-categorial iff $T$ has a model which belongs to $\mathcal{M}$. Rosenstein also showed that if $T$ is $\omega$-categorial then it is finitely axiomatizable; (ii) extends this result. Läuchli and Leonard in [4] define another class of models $\mathcal{S}'$ such that $\mathcal{S}' \subseteq \mathcal{S}'$. They replace the operation (3) by the operations

4. $\omega(\mathcal{A}) = \mathcal{A} \cdot \omega$

5. $\omega^*(\mathcal{A}) = \mathcal{A} \cdot \omega^*$

and define $\mathcal{N}$ as the smallest class which is closed under (1), (2), (4), and (5). They prove that every sentence which is true in some linearly ordered set is also true in some model which belongs to $\mathcal{N}$. So they conclude that every complete theory which is finitely axiomatizable has a model which belongs to $\mathcal{N}$. However it is not true that the complete theory of every model in $\mathcal{N}$ is finitely axiomatizable, (take for instance $\omega + \omega^*$).

In Section 7 we show that if $T$ is a complete theory of linear order with not more than $\omega$ unary predicates and $\| S_1(T) \| \leq \omega$ then $\| S_n(T) \| \leq \omega$ for every $n$. Indeed, the difficulty is in going from $S_1(T)$ to $S_2(T)$. Another question of the same nature is whether the statement that $F_1(T)$ is atomic implies that $F_2(T)$