SOME BALANCED COMPLETE BLOCK DESIGNS

BY

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ABSTRACT

Let $q = 6t \pm 1$, $v = 2q + 2$. The $(v/3)$ triples on $v$ marks may be partitioned into $q$ sets, each forming a BIBD of parameters $(v, 3, 2)$. Related results, some of them known, are also discussed briefly.

1. Notation and terminology

$q$ is a natural integer with $(q, 6) = 1$, that is, of the form $6t \pm 1$. $G$ is an additively written abelian group of order $q$, whose elements will be denoted by $0$ and by lower case Latin letters, except $x$ and $v$, with or without subscript. $H = G \oplus \mathbb{C}(2)$; the generator of $\mathbb{C}(2)$ is denoted by $\bar{0}$, and for $g \in G$, $g + \bar{0}$ will be written $\bar{g}$. $V$ is a set consisting of the elements of $H$ plus two additional marks $x$ and $y$. $|V| = v = 2q + 2$. Terminology will follow rather closely that of Hall [2, mainly Sect. 15.3]. Thus, we shall attempt a mixed difference construction. Specifically, in Section 3 we shall select a set $S(0)$ of triples on the elements of $V$, being a BIBD of parameters $(v, 3, 2)$. This will form the first layer (corresponding to a block in the mixed difference procedure described in [2]). Then, for every $q \in G$, we shall obtain in Section 4 a new layer $S(g)$, by adding $g$ to every element of $H$ appearing in a triple of $S(0)$. These $q$ BIBDs will have no triple in common and exhaust, among them, all $v \choose 3$ triples on $V$. Taking such a set of $q$ BIBDs as a single structure, with $G$ acting as group of automorphisms, the designation of complete block design for it would appear justified. We shall then sketch, in Section 6, a similar partition of triples on a set into layers covering the pairs once (Steiner triples); next, of pairs into layers covering the singletons once (see (7.1), Matchings) or twice (see (7.2), Hamiltonian cycles), as was pointed out, in this context, by

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Simmons [5]. Lastly, a somewhat illegal example of partitioning the \((t+1)\)-tuples of an infinite set into layers covering the \(t\)-tuples once.

2. Some more terminology and preliminary construction

Let \(\tau\) denote the automorphism of \(G\) mapping each \(g \in G\) into \(-2g\). Then \(0\tau = 0\) and since \(G\) can contain no element of order 3 (as \((q, 3) = 1\), \(g\tau \neq g\) for \(g \neq 0\). Thus \(\tau\) permutes the elements of \(G^\#(G^\# = G \setminus \{0\})\) in cycles. Call the ordered pair \((g, g\tau), g \in G^\#\), an arc and \((-g, -g\tau)\) the opposite arc. There are thus \(\frac{1}{2}(q-1)\) pairs of opposite arcs. Such pairs of arcs may lie on the same cycle (which is then of even length) or not (and then, sequences of such arcs form pairs of opposite cycles). Both situations may occur within the same group, as illustrated by \(G = C(35)\), say.

**Step 0.** Of each pair of opposite arcs (on the same cycle or not) colour arbitrarily one red, one blue. (This may be done in \(2^{q-1/2}\) ways.)

3. Construction of \(S(0)\)

**Step 1a.** Take each triple of different elements \((a, b, c), a, b, c \in G\), if \(a + b + c = 0\).

**Step 1b.** For each triple thus formed, add 7 more, independently replacing \(a, b, c\) by \(\bar{a}, \bar{b}, \bar{c}\) respectively.

**Step 1c.** For each arc \((a, b)\) (that is, \(b = -2a\)), take the two triples \((a, b, \bar{a})\) and \((a, \bar{b}, \bar{a})\).

(So far, we have collected \(\frac{3}{2}(q-1)(2q-1)\) triples, namely all the triples of different elements of \(H\) summing to 0 or \(0\); and, except for pairs of elements of \(H\) corresponding to pairs forming an arc in \(G\), each pair of elements of \(H\) has been covered twice).

**Step 2a.** If \((a, b)\) is a red arc, add the 4 triples \((x, a, b), (x, \bar{a}, \bar{b}), (y, a, \bar{b}), (y, \bar{a}, b)\).

**Step 2b.** If \((a, b)\) is a blue arc, add the 4 triples \((y, a, b), (y, \bar{a}, \bar{b}), (x, a, \bar{b}), (x, \bar{a}, b)\).

**Step 3.** Add the 4 triples \((x, y, 0), (x, y, \bar{0}), (x, 0, \bar{0}), (y, 0, \bar{0})\). Thus, each pair of elements of \(V\) has been covered exactly twice: the pairs originating from arcs have been covered a second time in Step 2 and for \(g \in G^\#\), each pair \((x, g), (x, \bar{g})\), \((y, g), (y, \bar{g})\) has also been covered twice, once for \(g\) being the first element of an