On the Dynamical Symmetry of the Nonisotropic Oscillators.

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It is well known that the isotropic harmonic oscillator with \( n \) degrees of freedom has a dynamical symmetry group \( SU_n \). It has been stated (1) that the same dynamical group belongs to any \( n \)-dimensional oscillator with frequencies at rational ratios. This is generally not correct for \( n > 2 \) even when the frequencies are at integral ratios, except in particular cases.

1. Degeneracy of the levels.

To decide in the simplest way if the oscillator

\[
H = \sum_{k=1}^{n} \frac{1}{2m} \left\{ p_k^2 + (m \omega_k x_k)^2 \right\},
\]

where \( \alpha_k = 1 \) and \( \alpha_k \) are integers, may belong to the \( SU_n \) group, it is enough to compare the order of degeneracy of the energy levels of the system with the dimensions of the tensor representations of that group.

It can be checked that the number of linearly independent eigenstates belonging to the following level:

\[
E_N = \left\{ N + \frac{1}{2} \sum_{i=1}^{n} \alpha_i \right\} \hbar \omega
\]

\((N = 0, 1, 2, \ldots)\)

is given by the formula

\[
\sum_{\lambda} \sum_{j_{n-1}=1}^{\lambda_{n-1}} \ldots \sum_{j_1=0}^{\lambda_1} \left( \lambda_{n-1}, \ldots, j_1, j_0 + 1 \right),
\]

(1)

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with

\[ \lambda = \left[ \frac{N}{\alpha_n} \right], \quad \lambda_{j_n} = \left[ \frac{N - j_n \alpha_n}{\alpha_{n-1}} \right], \ldots, \quad \lambda_{j_2, j_1} = \left[ \frac{N - \sum_{i=3}^{n} j_i \alpha_i}{\alpha_2} \right], \]

where \([\ ]\) stands for \(\int\) integral part of \(\).

The dimensions of the irreducible representations of \(SU_n\) do not generally correspond to it. For example, even in the simple case \(n = 3, \alpha_2 = 1, \alpha_3 = 2\), the \(N = 1\) level is twofold degenerate while \(SU_3\) has no bidimensional irreducible representations.

2. - Particular cases.

However, if we consider the particular case when \(\alpha_2 = \ldots = \alpha_n = \alpha\) formula (1) becomes

\[ \sum_{j_{n-3}} j_{n-3} \sum_{j_{n-4}} j_{n-4} \ldots \sum_{j_3} j_3 \left\{ \left[ \frac{N}{\alpha} \right] - j_3 - \ldots - j_4 + 1 \right\}. \]

But these are exactly the dimensions of the fully symmetric tensor representations \([\lfloor N/\alpha \rfloor, 0, \ldots, 0]\) of \(SU_n\).

The realization of this only family of irreducible representations was to be expected owing to the dependence of energy on only one quantum number.

3. - The construction of generators.

Let

\[ a_K = \left( \frac{m_0 \omega x_K}{2\hbar} \right)^{\frac{1}{2}} q_K + \frac{i}{(2m_0 \omega x_K)^{\frac{1}{2}}} p_K \]

\((K = 1, \ldots, n)\)

be the usual creation and destruction operators.

Then we can construct the operators

\[ B_1^{(p)} = \frac{1}{\sqrt{\alpha}} \left\{ a_1 - \frac{1}{(a_1 a_1)^{\frac{1}{2}}} \right\}^{3-q-1} a_1 \left( \frac{1}{(a_1 a_1)^{\frac{1}{2}}} \right)^q \]

for each \(0 < \varphi < \alpha\).

Let us consider now the projection operators on the states having \(N = \nu x + \varphi\), when \(\nu\) is an integer:

\[ P^{(q)} = \left( \frac{\sin \left( \pi \left( a_1^{\dagger} a_1 + \alpha \sum_{K=2}^{n} a_K^{\dagger} a_K - \varphi \right) \right)}{\alpha \sin \left( \pi/\alpha \left( a_1^{\dagger} a_1 + \alpha \sum_{K=2}^{n} a_K^{\dagger} a_K - \varphi \right) \right)} \right)^2. \]