TWO COMMENTS ON DVORETZKY'S SPHERICITY THEOREM

BY
E. G. STRAUS

ABSTRACT
For any two positive integers k, l and any ε > 0 there exists an N(k, l, ε) so that given any l convex bodies C_1, ..., C_l symmetric about the origin in E^n with n ≥ N there exists a subspace E^k so that each C_i intersects E^k, or has a projection into E^k, in a set which is nearly spherical (asphericity < ε). The measure of the totality of E^k which intersect a given body in E^n in a nearly ellipsoidal set is considered and an affine invariant measure is introduced for that purpose.

A convex set C in E^n which is centrally symmetric about the origin is said to have asphericity
\[ \alpha(C) = 1 - \min_{x \in \text{bd}C} \frac{\|x\|}{\max_{x \in \text{bd}C} \|x\|} \]
where bdC is relative to the subspace spanned by C. Dvoretzky [1] proved that: For every positive integer k and every ε > 0 there exists a number N(k, ε) (e.g., N(k, ε) = exp(2^{15} ε^{-2} k^2 \log k)), so that for n ≥ N, every convex body (compact convex set with non-empty interior) in E^n which is symmetric about the origin there exists a subspace E^k with \( \alpha(C \cap E^k) < \varepsilon \).

In a recent paper [2] Dvoretzky remarks that the same result holds if we consider the projection C|E^k of C into E^k instead of C \cap E^k, since
\[ \alpha(C|E^k) = \alpha(C^* \cap E^k) \]
where C^* is the polar body of C. However he states as an unsolved question whether there is an \( N'(k, \varepsilon) \) so that for n ≥ N' there exists an E^k for which both
\[ \alpha(C \cap E^k) < \varepsilon \quad \text{and} \quad \alpha(C|E^k) < \varepsilon. \]

To give an affirmative answer to this question we prove the following.

THEOREM. For each pair of positive integers k, l and every ε > 0 there exists an N(k, l, ε) so that for n ≥ N and any l-tuple of convex bodies C_1, ..., C_l in E^n symmetric about the origin, there exists a subspace E^k so that
\[ \alpha(C_i \cap E^k) < \varepsilon \quad \text{for} \quad i = 1, ..., l. \]

Here \( N(k, 1, \varepsilon) = N(k, \varepsilon) \) and \( N(k, l + 1, \varepsilon) \leq N(N(k, l, \varepsilon), \varepsilon) \).

Received February 6, 1964.
Proof. For \( l = 1 \) this is Dvoretzky's theorem. Assume the theorem true for \( l \). Then for \( n \geq N(N(k,l,\varepsilon),\varepsilon) \) there exists an \( \varepsilon_n \in E^{N(k,l,\varepsilon)} \) so that \( \varepsilon(C_1 \cap E^{N(k,l,\varepsilon)}) < \varepsilon \) and by the induction hypothesis applied to \( C'_i = C_i \cap E^{N(k,l,\varepsilon)} \), \( i = 2, \ldots, l + 1 \); there exists an \( \varepsilon_i \in E^{N(k,l,\varepsilon)} \) so that \( \varepsilon(C'_i \cap E^{N(k,l,\varepsilon)}) < \varepsilon \) for \( i = 2, \ldots, l \). On the other hand we have \( \varepsilon(C_1 \cap E^{N(k,l,\varepsilon)}) \leq \varepsilon(C_1 \cap E^{N(k,l,\varepsilon)}) < \varepsilon \) so the result holds with \( N(k,l,\varepsilon,\varepsilon) = N(N(k,l,\varepsilon),\varepsilon) \).

Dvoretzky's question is now answered in the affirmative for \( C_1 = C, C_2 = C^* \), \( l = 2 \). The bound computed here grows very rapidly since it involves \( l \)-fold iteration of an already very rapidly increasing function of \( k \) and \( 1/\varepsilon \).

A second question raised in [2] can be answered in the negative. Dvoretzky proves that it is not possible to give a uniform positive lower bound for the Haar measure of the set of all \( k \)-planes \( (k \geq 2) \) which intersect a convex body \( C \) in \( E^n \) in a set of asphericity \( \varepsilon \). His example is an ellipsoid of revolution with a very large axis on its axis of revolution. He asks therefore whether such a uniform lower bound could exist if asphericity is replaced by unellipsoidality, that is the minimum asphericity of all affine transforms of the set.

As an example of a body for which this is not the case we consider the union of two spherical caps:

\[
x_1^2 + \ldots + x_{n-1}^2 + (x_n - 1 + \delta)^2 \leq 1, \ x_1^2 + \ldots + x_{n-1}^2 + (x_n + 1 - \delta)^2 \leq 1; \ 0 < \delta < 1.
\]

Every \( E^2 \) intersects \( C \) in a lens, which in terms of Cartesian coordinates \((y_1,y_2)\) on \( E^2 \) can be given by

\[
y_1^2 + (y_2 - r + \delta')^2 \leq r^2, \ y_1^2 + (y_2 + r - \delta')^2 \leq r^2.
\]

Here

\[
r^2 = 1 - (1 - \delta)^2 + (r - \delta')^2
\]

\[
r - \delta' = (1 - \delta)\cos \gamma
\]

where \( \gamma \) is the angle between \( E^2 \) and the \( x_n \)-axis. Thus for \( \delta \) sufficiently small we have \( \delta'/r \) arbitrarily small for all \( \gamma \) outside an arbitrarily small neighborhood of \( \pi/2 \). Thus all we need is the following.

Lemma. The lens

\[
x^2 + (y - 1 + \delta)^2 \leq 1, \ x^2 + (y + 1 - \delta)^2 \leq 1; \ 0 < \delta < 1
\]

has unellipsoidality \( > 1/10 + O(\sqrt{\delta}) \).

Proof. Because of the symmetry of the lens it suffices to consider diagonal transformations of the form \( x' = x, y' = cy \). The radius in the \( x \)-direction remains \( \sqrt{2\delta} + O(\delta) \) while the radius in the \( y \)-direction becomes \( c\delta \). Thus, if the unellipsoidality is \( \leq 1/10 + O(\sqrt{\delta}) \) we have \( c\delta \leq (10/9)\sqrt{2\delta} + O(\delta) \). Now the point \((\sqrt{\delta},\delta/2 + O(\delta))\) on the lens goes into \((\sqrt{\delta},c\delta/2 + O(\sqrt{\delta})\) whose distance from the origin is