SINGULAR SEMI-LINEAR EQUATIONS
IN $L^1(\mathbb{R})$

BY

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ABSTRACT

Let $g$ be a positive continuous function on $\mathbb{R}$ which tends to zero at $-\infty$ and which is not integrable over $\mathbb{R}$. The boundary-value problem $-u'' + g(u) = f$, $u'(\pm \infty) = 0$, is considered for $f \in L^1(\mathbb{R})$. We show that this problem can have a solution if and only if $g$ is integrable at $-\infty$ and if this is so then the problem is solvable precisely when $\int_{-\infty}^{-\infty} f(t) dt > 0$. Some extensions of this result are also given.


\[
\begin{align*}
- u''(x) + \beta(u(x)) &= f(x), \quad -\infty < x < \infty \\
\beta'(\pm \infty) &= 0 \\
\beta'' &\in L^1(\mathbb{R})
\end{align*}
\]

(\ast)

has a solution for each $f \in L^1(\mathbb{R})$ with $\int_{-\infty}^{\infty} f > 0$ if (and only if) $\beta$ is integrable at $-\infty$. Here $\beta$ is a given positive monotone increasing continuous function on $\mathbb{R}$. In fact, they discuss the more general situation when $\beta$ is a maximal monotone graph. In this paper we consider several extensions of the problem (\ast) and provide another technique for proving that these equations have a solution. In particular, we recover the result of Crandall and Evans by different means.

**Theorem 1.** Let $g$ be a positive continuous function on $\mathbb{R}$ with

$$
\lim_{t \to -\infty} g(t) = 0, \quad \int_{-\infty}^{\infty} g(s) ds \quad \text{divergent}.
$$

Let $L^1_+ = \{f \in L^1(\mathbb{R}); \int_{-\infty}^{\infty} f > 0\}$; for $f \in L^1_+$ consider the problem

$*$ Sponsored by the United States Army under Contract No. DAAG29-75-C-0024 and by the National Science Foundation, Grant MPS 75-05501.

Received August 2, 1976
The following are equivalent:
(a) (1) has a solution for all \( f \in L^1 \),
(b) (1) has a solution for some \( f \in L^1 \),
(c) \( g \) is integrable at \(-\infty\).

PROOF. (a) implies (b) is trivial. To see that (b) implies (c) suppose there is a function \( u \) with \( u'' \in L^1 \), \( u'(-\infty) = 0 \), and

\[
- u''(x) + g(u(x)) = f(x), \quad -\infty < x < \infty
\]

(1)

\[
\begin{align*}
&\quad u'' \in L'(\mathbb{R}) \\
&\quad u'(\pm \infty) = 0.
\end{align*}
\]

for some \( f \in L^1 \). Then \( u' \in L^\infty \) and \( u \) tends to \(-\infty\) at both \( \pm \infty \) for the following reason. Suppose there is a sequence \( x_n \to \infty \) with \( \lim u(x_n) = L > -\infty \). Let \( \{y_n\} \) be any other sequence of real numbers tending to \( +\infty \). Then from (2) we get

\[
-\frac{1}{2}(u'(y_n))^2 + \frac{1}{2}(u'(x_n))^2 + H(u(y_n)) - H(u(x_n)) = \int_{x_n}^{y_n} fu'
\]

where

\[
H(t) = \int_0^t g(s)ds.
\]

Hence, \( \lim H(u(y_n)) \) exists and equals \( H(L) \). Thus, \( H(u(t)) \) has a limit at \( \infty \) which implies that \( u \) has limit \( L \) at \( \infty \) since \( H \) is strictly monotone. But then \( g(u(t)) \) tends to \( g(L) > 0 \) as \( t \to \infty \) which contradicts the fact that \( g(u(t)) \) is in \( L'(\mathbb{R}) \). An identical argument shows \( u \) tends to \(-\infty\) at \(-\infty\). With \( H \) as above we also have

\[
\frac{1}{2}[(u'(y))^2 - (u'(0))^2] + H(u(0)) - H(u(y)) = \int_0^y fu'
\]

for each \( y, y < 0 \). Thus, \( H(u(y)) \) has a finite limit as \( y \to -\infty \). Since \( u(y) \to -\infty \) as \( y \to -\infty \) we find that \( H(s) \) has a finite limit as \( s \to -\infty \) implying that \( g \) is integrable at \(-\infty\). The proof that (c) implies (a) is the most difficult. The first step is to show that the set of those \( f \in L^1 \) for which (1) is solvable is closed in \( L^1 \); the second step is then obviously to show that the set of those \( f \in L^1 \) for which (1) is solvable is dense in \( L^1 \). To prove the first assertion, let \( f_n \to f \) in \( L'(\mathbb{R}) \), with \( f, f_n \in L^1 \). Let \( u_n \) satisfy