EVERY PLANAR GRAPH HAS
AN ACYCLIC 7-COLORING

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ABSTRACT
A proper coloring of the vertices of a graph is said to be acyclic provided that no cycle is two colored. We prove that every planar graph has an acyclic seven coloring.

1. Introduction

A k-coloring of the vertices of a graph is an assignment of one of the colors 1, 2, \ldots , k to each vertex so that no two adjacent vertices receive the same color. A k-coloring is said to be acyclic if the subgraph induced by the vertices colored with any two colors has no cycle. The acyclic chromatic number of a graph G, denoted by \( a(G) \), is the minimum value of \( k \) for which \( G \) has an acyclic \( k \)-coloring. The purpose of this paper is to prove the following:

**Theorem.** If \( G \) is planar, then \( a(G) \leq 7 \).

In [3], Branko Grünbaum conjectured that if \( G \) is planar, \( a(G) \leq 5 \). He proved that if \( G \) is planar, \( a(G) \leq 9 \). John Mitchem [6] proved that if \( G \) is planar, \( a(G) \leq 8 \). In a forthcoming paper [5], A. V. Kostochka proves, using methods different than ours, that if \( G \) is planar, \( a(G) \leq 6 \). An analog to the above theorem for toroidal graphs is presented in [1].

The proof is by induction on the number of vertices of \( G \). The theorem is trivially true if \( G \) has seven or fewer vertices.

A graph \( H \) is said to be reducible if, for any graph \( G \) having \( H \) as a subgraph, there exists a graph \( G' \), having fewer vertices than \( G \), and having the property that any acyclic 7-coloring of \( G' \) can be extended to an acyclic 7-coloring of \( G \). In section II we present a list of reducible graphs.

Now, suppose \( G \) is a planar graph on the fewest vertices such that \( a(G) = 8 \).

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We assume $G$ is a triangulation; if not, edges may be added to make it so without reducing the acyclic chromatic number. In section III we show that every triangulation of the plane (thus, in particular, $G$) must contain a reducible graph. This completes the proof of the theorem; for $G$, being a minimum counter-example, can contain no reducible graph.

II. The reducible graphs

**Lemma 1.** Let $G$ be a planar graph on the fewest vertices such that $a(G) = 8$. $G$ can have no 3-cycle whose interior and exterior are both non-empty.

**Proof.** Suppose that $a, b, c$ is such a 3-cycle. Since the subgraph consisting of the cycle together with its interior has fewer vertices than $G$, it can be acyclically 7-colored. Assume that $a, b$ and $c$ are colored 1, 2 and 3, respectively.

Likewise, the subgraph consisting of the cycle together with its exterior can be acyclically 7-colored. We may permute the colors in this subgraph so that $a, b$ and $c$ are again colored 1, 2 and 3, respectively. Now this coloring of the exterior vertices may be adjoined to the coloring of the interior vertices to give a 7-coloring of $G$. A two-color cycle in $G$ can only arise from a two-color cycle in either the interior or the exterior subgraph.

**Corollary.** A vertex of degree three is reducible.

**Lemma 2.** A vertex of degree four adjacent to a vertex of degree less than seven is reducible.

**Proof.** Suppose $x$ is a vertex of degree four with neighbors labeled, clockwise, $y, a, b$ and $c$, with $y$ having degree less than seven. Form $G'$ by deleting $x$ and adding an edge connecting $a$ and $c$. Acyclically 7-color $G'$, say with $a, b$ and $c$ colored 1, 2 and 3, respectively.

Transfer this coloring to $G$, with $x$ left uncolored. If $y$ is not colored 2, then $x$ can be made any color not used by $a, b, c$ or $y$. If $y$ is colored 2, then $x$ can be made whichever of the colors 4, 5, 6, 7 that is not used by the remaining three, or fewer neighbors of $y$ (i.e. other than $a, c$ and $x$).

All of the remaining reducible graphs must avoid a certain type of 4-cycle. A 4-cycle is said to be fat if both the interior and the exterior of the cycle contain more than one point. (We consider the interior to be the region with fewer vertices.)

**Lemma 3.** Suppose $H$ consists of a vertex $x$ of degree five with neighbors labeled clockwise $y, a, b, c$ and $d$. If $y$ is of degree five or six and if $a, x$ and $d$ are not on any fat 4-cycle, then $H$ is reducible.