Correct Asymptotic $\lambda$ Behavior of the Partial-Wave Amplitude.

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Summary. — The asymptotic $\lambda$ behavior of the analytically continued partial-wave amplitude is commonly given (incorrectly) as $O(\lambda^{-1} \exp [-\Gamma \lambda])$. This result is exorcised by a careful study of the past Regge literature. The original (correct) result of $O(\lambda^{-1} \exp [-\Gamma \lambda])$ is resurrected, and the conditions under which it is valid, as well as its implications, are carefully discussed.

Since the advent of Regge poles in the early sixties, there has existed an almost universal assumption that the asymptotic $\lambda$ behavior ($\lambda = l + \frac{1}{2}$) of the partial-wave amplitude in the right half-plane is given by

$$\lim_{|\lambda| \to \infty} a(\lambda, k) \sim O(\lambda^{-1} \exp [-\Gamma \lambda]) .$$

In eq. (1) the limit is such that Re $\lambda > 0$ and $\Gamma > 0$ with $k$ being the momentum. In relativistic theory $k$ is replaced by $s$, the center-of-mass energy squared.

It is the purpose of this note to point out that the result (1) is incorrect, generally, for a superposition of Yukawa potentials except in a special case. It is also incorrect in relativistic theory except for a special case.

We will also illuminate the reasons leading to acceptance of the incorrect result (1), and we will give the correct limit as originally derived (and then ignored) by Bottino, Longoni, and Regge (1). This is

$$\lim_{|\lambda| \to \infty} a(\lambda, k) \sim O(\lambda^{-1} \exp [-\Gamma \lambda]) .$$

The importance of the difference between eqs. (1) and (2) for the Sommerfeld-Watson background integral will also be discussed.

It might appear that the small difference between eqs. (1) and (2) is such as to hardly merit a notice. However, the incorrect result has become so firmly embedded in the physics literature of the past several years that clearly a large majority of physicists do not know the correct result. Many books, including ones by Chew (2), Newton (3), and de Alfaro and Regge (4), have quoted eq. (1) while numerous papers (5,6) also use the incorrect limit. In addition the convergence of the integrals (11) and (12) below are affected by the extra $\lambda^{-1}$ factor.

The source of this incorrect understanding may be simply illuminated by tracing the early literature. In ref. (1) the authors used a WBK method to derive the asymptotic form for $a(\lambda, k)$ in potential theory. Their eq. (3.21) gave the Born approximation for the phase shift which they showed to be exact in the ray limit $|\lambda| \to \infty$ for $|\arg \lambda| < \pi/2$ and $k$ fixed. This result is

$$
\delta_{\text{Born}} = -\frac{V_0}{2k} \int Q_1 \left(1 + \frac{\mu^2}{2k^2}\right) \sigma(\mu) \, d\mu,
$$

where $\sigma(\mu)$ is related to the potential by

$$
V(r) + V_0 \int \sigma(\mu) \frac{\exp[-\mu r]}{r} \, dr
$$

making eq. (4) a superposition of Yukawa potentials. In eq. (3) $Q_1$ is the Legendre function of the second kind.

They evaluated eq. (3) for large $|\lambda|$ and found in the ray limit ($|\arg \lambda| < \pi/2$)

$$
\lim_{|\lambda| \to \infty} \delta_{\text{Born}}^{\lambda^{-1}} \sim -k \sqrt{\frac{\pi}{2}} \frac{V_0 \sigma(m)}{2m} \exp[-\alpha \lambda] \left(\sinh \alpha\right)^{\frac{1}{2}},
$$

$$
\alpha = \cosh^{-1} \left(1 + \frac{m^2}{2k^2}\right).
$$

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(5) N. N. Khuri: *Phys. Rev.*, 132, 914 (1963) is an example of an early paper which uses assumption (1) without stating when it is valid.
(6) W. J. Abe and Y. N. Srivastava: *Nuovo Cimento*, 52 A, 537 (1967) is an example of a recent paper which incorrectly evaluates the same integral, obtaining eq. (1), which is done correctly in ref. (1), where eq. (2) is obtained.