EXAMPLES OF $\mathcal{L}_p$ SPACES ($1 < p \neq 2 < \infty$)

BY

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ABSTRACT

We present a simple method for constructing new $\mathcal{L}_p$ spaces ($1 < p \neq 2 < \infty$) out of old ones. Using this method and results of H. P. Rosenthal we prove the existence of a sequence of mutually nonisomorphic separable infinite dimensional $\mathcal{L}_p$ spaces ($1 < p \neq 2 < \infty$).

1. Introduction

Since the theory of $\mathcal{L}_p$ spaces was introduced in [3], [5] it was an open problem whether there exist infinitely many mutually nonisomorphic separable infinite dimensional $\mathcal{L}_p$ spaces ($1 < p \neq 2 < \infty$) (The case $p = 1$ was solved in [2]). The purpose of this paper is to solve this problem.

**Theorem.** There exists a sequence of mutually nonisomorphic, separable, infinite dimensional $\mathcal{L}_p$ spaces ($1 < p \neq 2 < \infty$).

The only knowledge required from the theory of $\mathcal{L}_p$ spaces is the fact proved in [3], [5] that $X$ is a separable $\mathcal{L}_p$ space ($1 < p \neq 2 < \infty$) if and only if it is isomorphic to a complemented subspace of $L_p(I)$ ($I = (0, 1)$) and it is not isomorphic to $l_2$.

The proof of the theorem is carried out by introducing a very simple method to construct new $\mathcal{L}_p$ spaces out of old ones and combining this method with results of H. P. Rosenthal [9] concerning the span in $L_p$ of a sequence of independent random variables.

The notations are standard and those which are not explained here can be found in [6].

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* This is part of the author's Ph. D. thesis written at the Hebrew University of Jerusalem under the supervision of Professor J. Lindenstrauss. The author wants to thank Professors J. Lindenstrauss and L. Tzafriri for their interest and for helpful discussions concerning the material of this paper.

Received April 14, 1975
We only mention that $(r_i)_{i=1}^\infty$ is the sequence of the Rademacher functions. 

$I$ denotes the unit interval.

The symbol $\approx \ (\approx)$ denotes inequalities in both sides with constants which do not depend on the scalars (respectively $-$ with constants $K$ and $K^{-1}$).

2. Preliminaries

DEFINITION. Let $(X_i)_{i=1}^n$ be a finite sequence of subspaces of $L_p(I)$ $(1 \leq p < \infty)$; we define $X_1 \otimes X_2 \otimes \cdots \otimes X_n = \bigotimes_{i=1}^n X_i$ to be the closed linear span in $L_p(I^n)$ of functions of the form $(x_1 \otimes \cdots \otimes x_n)(t_1, t_2, \cdots, t_n) = x_1(t_1) \cdot x_2(t_2) \cdots x_n(t_n); x_i \in X_i$. (This definition coincides with the completion of the usual tensor product in a certain norm.)

LEMMA 1. Let $X_i, i = 1, 2, \cdots, n$ be complemented subspaces of $L_p(I)$ $(1 \leq p < \infty)$; then $\bigotimes_{i=1}^n X_i$ is complemented in $L_p(I^n)$.

LEMMA 2. Let $X_i, Y_i, i = 1, 2, \cdots, n$, be subspaces of $L_p(I)$ and let $T_i : X_i \rightarrow Y_i$ be isomorphisms onto, $i = 1, 2, \cdots, n$. Then $\bigotimes_{i=1}^n T_i : \bigotimes_{i=1}^n X_i \rightarrow \bigotimes_{i=1}^n Y_i$ is an isomorphism.

LEMMA 3. Let $X_i, i = 1, 2, \cdots, n$, be subspaces of $L_p(I)$ $(1 \leq p < \infty)$ with unconditional bases $(x_i^j)_{j=1}^\infty, i = 1, 2, \cdots, n$ respectively. Then $(x_1^1 \otimes x_2^1 \otimes \cdots \otimes x_n^1)_{i=1}^{\infty}$ constitutes an unconditional basis for $\bigotimes_{i=1}^n X_i$.

Lemmas 1 and 2 are well known and easy to prove. Let us just mention that if, for example, $n = 2$ and $P_i : L_p(I) \rightarrow X_i, i = 1, 2$, are the given projections in Lemma 1 and if $k(u, v)$ is a continuous function then the projection of $k$ into $X_1 \otimes X_2$ is given in the following way. Fix $v$ and let $h(\cdot, v)$ be $P_2(k(\cdot, v))$. Now consider $h(u, v)$ as a representing function of its equivalent class; then for almost every $u \in I$, $h(u, \cdot) \in L_p(I)$. Apply $P_2$ to this function.

PROOF OF LEMMA 3. The proof is carried out by induction. We shall consider only the case $n = 2$. The induction step is carried out in a similar manner.

It is obvious that span $[(x_1^1 \otimes x_2^1)_{i=1}^\infty] = X_1 \otimes X_2$. Now by the unconditionality of $(x_i^i)_{i=1}^\infty, (x_j^j)_{j=1}^\infty$, by the generalization of Khinchine's inequality for expressions of the form $\int_0^1 \int_0^1 | \sum_{j=1}^n b_{ij} f_i(t) f_j(s) |^p dt ds$ and by Fubini's theorem we get that for all scalars $(a_{ij})_{i=1}^\infty$.