ON CO-\(\kappa\)-SOUSLIN RELATIONS†

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ABSTRACT
This is a continuation of Harrington and Shelah [3]; however, the contents of this paper are self-contained.

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[We prove: if \(\leq\) is a co-\(\kappa\)-Souslin relation which defines a quasi-linear order on \(\mathbb{R}\) (even after adding a Cohen real) then either there is a perfect set of pairwise disjoint intervals, or \((\mathbb{R}, \leq)\) has a dense subset of power \(\leq \kappa\)]

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§1. On the density of linear ordering

THEOREM. Suppose \(P\) is a co-\(\kappa\)-Souslin relation (on \(\mathbb{R}\)) which is a linear order (so we shall denote it by \(\leq\)) even after adding a Cohen real.

Then either \((\mathbb{R}, \leq)\) has a dense subset of power \(\leq \kappa\) or there is a perfect set of pairwise disjoint intervals.

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REMARK. In summer 1979, Friedman and Shelah (see [1]) proved this for $P$ a Borel relation. Shelah proved that $(\mathbb{R}, \leq)$ cannot be Souslin: If it is, by forcing by the set of intervals, we made it to have a strictly decreasing sequence of intervals $(a_i, b_i)$ ($i < \omega_1$). So $\{(a_{3i}, a_{3i+2}) : i < \aleph_1\}$ is a set of $\aleph_1$ pairwise disjoint intervals. But then, by the completeness theorem for $L_{\omega_1, \omega}(Q)$ (see Keisler [4]), this holds in the original universe (we use hereby the absoluteness). So assuming $(\mathbb{R}, \leq)$ has no countable dense sets, it has $\aleph_1$ pairwise disjoint intervals. This Friedman uses to prove the Theorem for $P$ Borel, adapting Harrington’s proof of Silver’s [6] theorem, using “there are $\aleph_1$ disjoint intervals” as a “bigness” property.

That proof does not seem to apply for the present theorem.

This proof uses the method of [3] (with choice) but is represented fully.

If you have difficulties, read §2 here and/or [3] and they may be explained in more detail there.

PROOF. First we choose by induction on $i < \kappa^+$, reals $a_i, b_i$ such that

(*) $a_i < b_i$, and for every $j < i$, $a_j < b_j \leq a_i < b_i$, or

$a_i < b_i < a_j < b_j$ or $a_i < a_j < b_j < b_i$

and

(**) if there are $\kappa^+$ pairwise disjoint (closed intervals) then $\{(a_i, b_i) : i < \kappa^+\}$ are such intervals (i.e. for $i < j$, $b_j < a_i$ or $b_i < a_i$).

Why can we do this? If we cannot choose $a_i, b_i$, let $A = \{a_j, b_j : j < i\}$, then in every Dedekind cut of $A$, there is at most one element of $\mathbb{R} - A$ [by the order $\subseteq$; more exactly, one equivalence class modulo $x \leq y \iff y \leq x$], so $(\mathbb{R}, \subseteq)$ has a dense subset of power $|2\mathbb{N}| \leq \kappa$.

Taking care of (***) is trivial.

Let $f: \kappa \rightarrow \text{the family of open subsets of } \mathbb{R} \times \mathbb{R}$ be such that

$$\neg xPy \equiv (\exists \eta \in \kappa) \land \bigwedge_{n<\omega} (x, y) \in f(\eta \restriction n).$$

Extend $\langle H(\kappa^+), \in \rangle$ by Skolem functions and get a model $\mathcal{C}$, and let $N < \mathcal{C}$ be a countable elementary submodel such that $g, h \in N$ where $g, h: \kappa^+ \rightarrow \mathbb{R}$, $g(i) = a_i$, $h(i) = b_i$.

We define a forcing notion ($t$ ranges over the rationals, $\bar{y}_i = \langle y_{ki} : i < \omega \rangle$, but the formula $\varphi$ involves only a finite initial segment)

$$Q = \{\varphi(x_0, \bar{y}_0, x_n, \bar{y}_n, \cdots, \bar{c}) :$$

$$\begin{align*}
&\text{ } t_0 < \cdots < t_n \text{ in } Q, \bar{c} \in N, \varphi \vdash \text{"\! } x_i \text{ an ordinal } \leq \kappa^+ \text{"}, \\
&\text{ and } \mathcal{C} \models \exists^* x_0 \exists^* \bar{y}_0 \exists^* x_n \exists^* \bar{y}_n \cdots \varphi, \end{align*}$$

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