AN ADDITIVITY PRINCIPLE FOR
GOLDIE RANK†

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ABSTRACT
Let A be a noetherian ring. In general A will not admit a classical Artinian ring of quotients. Yet a problem in enveloping algebras leads one to consider the possible embedding of A in a prime ring B which is finitely generated as a left and a right A module. Under certain additional technical assumptions, it is shown that the set S of regular elements of A is regular in B and is an Ore set in both A and B with \( S^{-1}A \) and \( S^{-1}B \) Artinian. This enables one to establish the following additivity principle for Goldie rank. Let \( \{P_1, P_2, \ldots, P_i\} \) be the set of minimal primes of A. Then under the above conditions it is shown that there exist positive integers \( z_1, z_2, \ldots, z_i \) such that

\[
\sum_{i=1}^{i} z_i \text{rk} (A/P_i) = \text{rk} B,
\]

where \( \text{rk} \) denotes Goldie rank. This applies to the study of primitive ideals in the enveloping algebra of a complex semisimple Lie algebra.

1. Introduction

Let \( g \) be a complex semisimple Lie algebra, \( U(g) \) its enveloping algebra, \( Z(g) \) the centre of \( U(g) \), \( \text{Prim } U(g) \) the primitive spectrum of \( U(g) \) and \( \pi: I \mapsto I \cap Z(g) \) the projection of \( \text{Prim } U(g) \) onto \( \text{Max } Z(g) \). The problem of classifying \( \text{Prim } U(g) \) and in particular the Jantzen conjecture [1], 5.9, motivate [6], 11.1 an additivity principle for Goldie rank.††Our main result, Theorem 3.9, establishes such a principle in a form suitable for application to enveloping algebras. Indeed

† This paper was written while the authors were guests of the Institute for Advanced Studies, The Hebrew University of Jerusalem. The first author was on leave of absence from the Centre Nationale de la Recherche Scientifique, France.

†† Also referred to as Goldie dimension, though the former terminology is more common in enveloping algebras.

Received February 14, 1978
when suitably combined with the methods of [6], it gives [7] a strong lower bound on the cardinality of each Prim \( U(g) \) fibre relative to \( \pi \). For \( g \) simple of type \( A_\ast \) (Cartan notation) this coincides with Duflo's upper bound [4], prop. 9, and establishes Jantzen's conjecture in the form described in [6], 10.3.

The content of this paper and its implications for the Jantzen conjecture were reported at a meeting on enveloping algebras at Oberwolfach, 5–10 February 1978. The authors would like to thank J. C. McConnell for his careful reading of the original manuscript.

2. Gelfand–Kirillov dimension and Artinian rings of quotients

2.1. Let \( F \) be a commutative field and \( A \) a finitely generated \( F \)-algebra with identity 1. Fix a finite dimensional generating subspace \( V \) of \( A \). For each \( k \in \mathbb{N}^+ \), let \( V^k \) denote the subspace of \( A \) spanned by the monomials \( v_1 v_2 \cdots v_k : v_i \in V \) and define a filtration \( A^0 \subset A^1 \subset A^2 \subset \cdots \), on \( A \) through \( A^0 = F, A^k = V^1 + V^2 + \cdots + V^k \). Given \( M \) a finitely generated left \( A \) module, let \( M^0 \) denote a finite dimensional generating subspace and define a filtration \( M^0 \subset M^1 \subset M^2 \subset \cdots \), on \( M \) through \( M^k = A^k M^0 \). Define the left Gelfand–Kirillov dimension \( d_A(M) \) of \( M \) through

\[
d_A(M) := \lim_{k \to \infty} \frac{\log \dim_F M^k}{\log k}.
\]

(In particular \( d_A(M) = -\infty \) if \( M = 0 \).)

It is elementary and well-known that \( d_A(M) \) does not depend on the choice of generating subspaces \( V, M^0 \), as is also the following

**Lemma.** Let \( M, M_1, M_2, \cdots, M \), be finitely generated left \( A \) modules, \( B \) an \( F \)-algebra containing \( A \) and finitely generated as a left \( A \) module. Then

(i) \( d_A(M) = \sup_i d_A(M_i) \), given \( M = M_1 + M_2 + \cdots + M_r \).

(ii) \( d_A(A) \geq d_A(M) \), for \( A \) considered as a left \( A \) module.

(iii) \( d_A(M_i) \geq d_A(M) \) for any subquotient \( M_i \) of \( M \).

(iv) \( d_A(L) \geq d_A(Lb) \) for any \( b \in B \) and any finitely generated \( A \) module \( L \) of \( B \). Equality holds if \( xb = 0; x \in L \) implies \( x = 0 \).

(v) If \( I \) is an ideal of \( A \) contained in \( \text{Ann} M \), then \( d_{A/I}(M) = d_A(M) \).

If \( M \) is a right \( A \) module we can similarly define the right Gelfand–Kirillov dimension \( d_A^r(M) \) of \( M \). As \( A \) will be fixed throughout and because of (v) we shall generally drop the subscripts.

For any \( F \)-algebra \( C \) and any subset \( T \) of \( C \) we let \( l_C(T) \) (resp. \( r_C(T) \)) denote