DIVISION ALGEBRAS OF DEGREE 4 AND 8 WITH INVOLUTION

BY

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ABSTRACT

We develop necessary and sufficient conditions for central simple algebras to have involutions of the first kind, and to be tensor products of quaternion subalgebras. The theory is then applied to give an example of a division algebra of degree 8 with involution (of the first kind), without quaternion subalgebras, answering an old question of Albert; another example is a division algebra of degree 4 with involution (*) has no (*)-invariant quaternion subalgebras.

§0. Introduction

In this paper, $F$ denotes an arbitrary field of characteristic $\neq 2$, and $R$ is a central simple $F$-algebra. An *involution* (of the first kind) of $R$ is an anti-automorphism of degree 2 fixing $F$. The classic reference on central simple algebras, with or without involution, is [1] (especially section X), and some positive general results were given in [4].

It is well known that $[R:F] = n^2$ for some $n$; $n$ is called the *degree* of $R$. Then, for some uniquely determined natural number $k$ and some division algebra $D$, we have $R = M_k(D)$, the algebra of $k \times k$ matrices with entries in $D$. If $\deg(R) = 2$, we call $R$ a *quaternion $F$-algebra*. In this case, there are elements $a_1, a_2$, in $R$ such that $0 \neq a_1^2 \in F$, $0 \neq a_2^2 \in F$, $a_1a_2 = -a_2a_1$ and $R = F + Fa_1 + Fa_2 + Fa_1a_2$; letting $a_i = a_i^2$, we denote $R$ by the symbol $(a_1, a_2; F)$. $R$ has an involution (*), given by

$$(\gamma_1 + \gamma_2a_1 + \gamma_3a_2 + \gamma_4a_1a_2)^* = \gamma_1 - \gamma_2a_1 - \gamma_3a_2 - \gamma_4a_1a_2.$$
Conversely, given $\alpha_1, \alpha_2$ in $F$, we can construct $(\alpha_1, \alpha_2; F)$ by taking formal elements $a_i, i = 1, 2$, and defining $R = F + Fa_1 + Fa_2 + Fa_1a_2$, with multiplication induced by the rules $a_i^2 = \alpha_i$ and $a_1a_2 = -a_2a_1$.

Any tensor product (over $F$) of $m$ quaternion $F$-algebras is of degree $2^m$, and has an involution given by the tensor product of the respective involutions. On the other hand, any central division algebra $D$ with involution has degree $2^m$ for some $m$, and Albert [1] showed $D$ is a tensor product of quaternion subalgebras when $m = 2$. The following famous question thus arises:

**Question A.** If a central simple $F$-algebra of degree $2^m$ has an involution, is it isomorphic to a tensor product of quaternion $F$-algebras?

It suffices to consider only division algebras, and the first stage to consider is $m = 3$. A related question of interest is

**Question B.** Suppose $R$ is a tensor product of quaternion algebras and has a given involution $(\ast)$. Is $R$ then a tensor product of (possibly different) $(\ast)$-invariant quaternion subalgebras?

The main object of this paper is to give a negative answer to Question A; we have a division algebra $D$ of degree 8 with involution, which is not a tensor product of quaternions. It is noteworthy that by a theorem of Tignol [5], $M_2(D)$ is a tensor product of quaternions. (Our construction also works for $m \geq 3$.)

We also provide a counterexample to Question B, of degree 4. Namely, there is a division algebra $Q_1 \otimes Q_2$ with involution $(\ast)$ without any $(\ast)$-invariant quaternion subalgebra. (In this case, the involution must be of orthogonal type, by [4, theorem B]). The method is to study abelian crossed products (cf. [2]), giving necessary and sufficient conditions for an involution to exist. Comparing these criteria with the structure of tensor products of quaternion algebras, applied to “generic abelian crossed products”, we arrive at the counterexamples.

§1. Tensor products of quaternions

Let $R$ be a central simple algebra with a center $F$. A set of elements $S = \{r_i\}$ is called a quaternion generating set, or in short a $q$-generating set, if:

1. $0 \neq r_i \in F$,
2. $r_ir_i = \pm r_i r_i$. 