UNIQUE ERGODICITY
OF THE HOROCYCLE FLOW:
VARIABLE NEGATIVE CURVATURE CASE

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ABSTRACT
H. Furstenberg showed that horocycle flows on compact manifolds of constant negative curvature are uniquely ergodic. This paper generalizes his result to the case of variable negative curvature, in the more general context of flows whose orbits are the unstable manifolds of certain Anosov flows.

0. Introduction

Horocycles were introduced to dynamical systems in the study of the dynamics of geodesic flows. For a compact connected orientable 2-manifold $N$ of negative curvature, a horocycle is an unstable manifold of the geodesic flow (i.e., the set of all points, in the unit tangent bundle $(T_1N)$ of $N$, which are backwards asymptotic with a given point under the action of the geodesic flow). Classically, horocycles were defined geometrically as certain curves in the universal covering manifold $\tilde{N}$ of $N$ ([6]). Our horocycles are related to the classical horocycles as follows: take one of our horocycles, project it into $N$, and then lift that to a curve in $\tilde{N}$; the result being a classical horocycle. (See [1, p. 30], [2].)

A continuous 1-parameter flow (on $T_1N$) whose orbits are the horocycles is called a horocycle flow. We will use the geodesic flow to study the dynamics of horocycle flows. In particular, we show ((3.6)) that horocycle flows are uniquely ergodic (i.e., they have a unique invariant Borel probability measure), provided the geodesic flow is of class $C^2$ (the $C^2$ assumption is apparently unnecessary; see (1.8)). This was proved in the case of constant negative curvature by Harry Furstenberg ([5]). His paper provided the motivation for this work, although our method is different.

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What we actually prove ((3.5)), which was suggested by Charles Pugh, is more general than (3.6), namely: continuous flows (called $W^u$ flows), whose orbits are the unstable manifolds of suitable (see (1.7) and (1.8)) Anosov flows, are uniquely ergodic. (The geodesic flow of manifolds of negative curvature is the classical example of an Anosov flow.) The crucial (but mild) assumption is that the $W^u$ flow be minimal. The same thing works just as well for flows whose orbits are stable manifolds. With our parametrization (2.1) of the $W^u$ flow the unique invariant measure is the one which maximizes entropy for the Anosov flow; this measure is constructed in [12].

We solved the analogous problem for Anosov diffeomorphisms in the more general context of Axiom A attractors ([11]). The results of this paper will be extended to the Axiom A context in a joint paper with Rufus Bowen ([4]), using another method.

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1. Anosov background

Let $M$ be a compact connected Riemannian manifold and $\{f_t\}$ an Anosov flow. This means that $\{f_t\}$ is a ($C^r$-) differentiable flow without fixed points and there is a ($Df_t$-)invariant continuous splitting of the tangent bundle $TM = E^u \oplus E^s \oplus E$, where $E$ is the line bundle tangent to the flow direction and $E^u$ and $E^s$ satisfy:

$$(1.0) \quad \exists \text{ constants } \alpha > 0, 0 < \mu < 1 \text{ such that for } t \geq 0: \text{ if } v \in E^u, \text{ then } \|Df_t(v)\| \leq \alpha t \|v\|, \text{ and if } v \in E^s, \text{ then } \|Df_t(v)\| \leq \alpha t \|v\|. \text{ Let } l = \dim E^u, k = \dim E^s."

We will need some facts from stable manifold theory. Let $d$ be the induced Riemannian metric on $M$. The unstable manifold

$$(W^u(x)) = \{ y \in M : \lim_{t \to +\infty} d(f_{-t}x, f_{-t}y) = 0 \}.$$

The stable manifold

$$(W^s(x)) = \{ y \in M : \lim_{t \to +\infty} d(f_{t}x, f_{t}y) = 0 \}.$$

The weak stable manifold