A general method of constructing block codes between Bernoulli shifts is introduced. This method generalizes an example of Boyle and Tuncel.

1. Introduction

Let $A$ and $B$ be finite sets. Consider the spaces $X = A^\mathbb{Z}$ and $Y = B^\mathbb{Z}$. Let $S$ be the left shift on $X$ and $T$ the left shift on $Y$. A $k$-block code is a map $\phi : A^k \to B$. It gives rise to an equivariant continuous map from $X$ to $Y$, namely

$$(\phi x)_i = \phi(x_i, \ldots, x_{i+k-1}), \quad \text{where } x = \{x_i\} i \in \mathbb{Z}.$$ 

It is known that every equivariant continuous map is a block map composed with a power of the shift $S$.

If $p$ is a probability measure on $A$, that is $p$ is a probability vector, and if $\mu = p^\mathbb{Z}$ is the product measure on $A^\mathbb{Z}$, then $S$ acting on the measure space is a Bernoulli shift (B.S.) denoted by $B(p)$. Let $q$ be a probability vector for $B(q)$ is a continuous factor of $B(p)$ if there exists a $k$-block code $\phi$ such that $p^\mathbb{Z}$ is carried onto $q^\mathbb{Z}$ by the homomorphism $\phi$. We shall say that $B(q)$ is a trivial factor of $B(p)$ if there is a map

$$f : A \to B \quad \text{such that } q(b) = \sum_{f(a) = b} p(a), \quad \forall b \in B,$$

i.e., the vector $q$ is a clustering of $p$. Obviously in such a case there exists a 1-block map from $B(p)$ onto $B(q)$.

This paper is a contribution to the study of continuous Bernoulli factors which was initiated in [3].

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If $B(q)$ is a continuous factor of $B(p)$ it is evident that $|A| \geq |B|$ since topological entropy decreases under taking factors. Suppose $|A| = |B|$. Considering metrical entropy we know that $h(p) \geq h(q)$. The case $h(p) = h(q)$ was settled in [3] and it was there proved that $q$ is a rearrangement of $p$, and therefore there are no non-trivial factors in such a case. The case $|A| = |B|$ and $h(p) > h(q)$ is impossible since by [2] a continuous equivariant map which preserves the topological entropy is finite to one. But a finite to one map preserves the metrical entropy.

In [3] it was proved that any continuous factor of the $n$-shift $B(1/n, \ldots, 1/n)$ is trivial. It was conjectured in [3] that there are no non-trivial continuous factors. However, in [1] an example was constructed of a non-trivial continuous factor (see §3).

In this paper a method of constructing factors for B.S.s is presented. This “tree method” gives us examples of a 2-state B.S. as a continuous factor of a 4-state B.S., also a 3-state B.S. as a factor of a 4-state B.S. We also indicate how to get $k$-block maps between B.S.s which do not admit $k$-1-block maps.

2. The tree method

A $k$-tree $T$ is a set of $k$-tuples $(i_1, \ldots, i_k)$ of natural numbers (called branches) subject to the following restrictions. There is a natural number $n(\emptyset)$ and $1 \leq i_1 \leq n(\emptyset)$ and for each $(i_1, \ldots, i_j)$, $1 \leq j < k$ which is an initial segment of a branch there exists a natural number $n(i_1, \ldots, i_j)$ and $1 \leq i_{j+1} \leq n(i_1, \ldots, i_j)$. Observe that for each $1 \leq j < k$ there is a $j$-tree associated with $T$, namely the $j$-tuples $(i_1, \ldots, i_j)$ which are the initial segments of branches of $T$. This tree be denoted by $T(j)$.

A probability tree is a tree together with a set of probability vectors \{p(i_1, \ldots, i_j)\}, $0 \leq j < k$ such that the number of components of $p(i_1, \ldots, i_j)$ is equal to $n(i_1, \ldots, i_j)$. Given a probability tree $T$ we associate with it the probability vector of $T$, $p(T)$, in the following way. $p(T)$ is labelled by the branches of $T$ and

$$p(i_1, \ldots, i_k)(T) = \prod_{j=0}^{k-1} P_{i_{j+1}}(i_1, \ldots, i_j).$$

Given two probability vectors $p$ defined on $A$, and $q$ defined on $B$, we say that $q$ is a clustering of $p$, denoted $q < p$, if there exists a map $f : A \to B$ such that

$$q_b = \sum_{f(a) = b} p_a, \quad \forall b \in B.$$