ON SUBSEQUENCES OF THE HAAR SYSTEM
IN C(\Delta)

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ABSTRACT
Spaces arising as spans of subsequences of the Haar system in C(\Delta) are studied. It is shown that for any compact metric space H there is a subsequence whose span is isomorphic to C(H), yet that subsequences exist whose spans are not \mathcal{L}_\infty spaces.

1. In [1], Gamlen and Gaudet showed that only two Banach spaces, \ell_p and L_p, arise as the spans of subsequences of the Haar System in L_p, 1 < p < \infty. In this paper we investigate spans of subsequences of the Haar system in \text{C}(\Delta), where \Delta denotes the Cantor set. The situation is somewhat different. In section 2 we show that for any compact metric space H, there is a subsequence of the Haar system whose span is isomorphic to C(H), while in section 4 we give an example of a subsequence whose span is not an \mathcal{L}_\infty space. In section 3 we show that for all A \subseteq \mathbb{N}, \{[\varphi_n]_n \in A\} contains c_0, and provide a sufficient condition for \{[\varphi_n]_n \in \Delta\} to contain an isomorph of \text{C}(\Delta).

Our notation is standard. If C is a subset of a Banach space X, we use \[C\] to denote the closed linear span of C. We write \(X \sim Y\) (resp. \(X \cong Y\)) to denote that \(X\) is isomorphic (resp. isometric) to \(Y\).

The Haar system is a monotone basis for \text{C}(\Delta) and may be defined as follows. Let \(\mathcal{B} = \{\Delta_{n,i} : n = 0, 1, \ldots; 0 \leq i < 2^n\}\) be a basis of clopen sets for the topology of \(\Delta\) such that

(i) \(\Delta_{0,0} = \Delta\),

(ii) \(\Delta_{n,i} \cap \Delta_{n,j} = \emptyset\) if \(i \neq j\),

(iii) \(\Delta_{n+1,2i} \cup \Delta_{n+1,2i+1} = \Delta_{n,i}\),

and define \(\varphi_0 = \chi_{\Delta_{0,0}}\), \(\varphi_{2^n+i} = \chi_{\Delta_{n+1,2i}} - \chi_{\Delta_{n+1,2i+1}}\) for \(n = 0, 1, \ldots; 0 \leq i < 2^n\).

Received October 20, 1977 and in revised form May 18, 1978
The author would like to thank the referee for his helpful comments. In particular, Theorem 2.1 below is somewhat stronger than the original theorem. The proof of Theorem 2.1 presented here, which is simpler than the original proof, is due to the referee.

2. In this section we show that for any compact metric space $K$, there is a subsequence of the Haar system whose span is isomorphic to $C(K)$. If $K$ is uncountable, this follows from Milutin's Theorem [4, p. 174], and if $K$ is countable, then $K$ is homeomorphic to a closed subset of the Cantor set.

**Theorem 2.1.** For any closed subset $K \subset \Delta$, there exists a subsequence $\{\varphi_n\}$ such that $X = \{\{\varphi_n\}\} \cong C(K)$ and $X$ is complemented in $C(\Delta)$ by a projection of norm one.

**Proof.** Let $R$ denote the restriction operator from $C(\Delta)$ onto $C(K)$, let $A = \{n: n = 0 \text{ or } \varphi_n \text{ is not constant on } K\}$, and let $X = \{\{\varphi_n\}_{n \in A}\}$. Then, if $n \neq m$, $(R\varphi_n)(R\varphi_m)$ is either 0 or $R\varphi_{\max(n,m)}$, and an induction argument shows that $(R\varphi_{2^m})^2 = R\chi_{\Delta_{k,1}}$. Thus $R(X)$ is a separating subalgebra of $C(K)$. By the Stone–Weierstrass theorem, $R|_X$ is an isometry onto $C(K)$. Furthermore, $X$ is complemented by the projection $P = (R|_X)^{-1}R$.

3. In this section we show that any space $X$ arising as the span of a subsequence of the Haar system contains an isomorph of $c_0$, and give a sufficient condition for $X$ to contain an isometric isomorph of $C(\Delta)$.

**Theorem 3.1.** If $X = \{\{\varphi_n\}\}$, then $X$ contains a subspace isomorphic to $c_0$.

**Proof.** We show that $\{\varphi_n\}$ contains a subsequence equivalent to either (i) the unit vector basis in $c_0$ or (ii) the basis $\{\hat{x}_n\}$ for $c$ defined by $x_n(i) = 0$, $i < n$ and $x_n(i) = 1$, $i \geq n$.

If $\{\varphi_n\}$ contains a subsequence $\{\psi_m\}$ of disjointly supported functions, then (i) holds. Otherwise, there is a subsequence $\{\psi_m\}$ such that $\text{supp } \psi_{m+1} \subset \text{supp } \psi_m$. We may assume, in fact, that $\text{supp } \psi_{m+1} \subset \psi_m^{-1}(1)$. Then for any scalar sequence $\{a_j\}_{j=1}^\infty$,

$$\left\| \sum a_j \varphi_j \right\| = \max_j \left\{ \left| \sum_{j=1}^i a_j \right| \right\}$$

$$\leq \max_j \left( \max \left\{ \left| \sum_{j=1}^i a_j \right|, \left| \sum_{j=1}^i a_j - a_j \right| \right\} \right)$$

$$= \left\| \sum a_j \varphi_j \right\|.$$