DOMINATED ESTIMATES OF CONVEX COMBINATIONS OF COMMUTING ISOMETRIES*

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ABSTRACT
The principal result of this paper is that the convex combination of two positive, invertible, commuting isometries of \( L_p(X, \mathcal{F}, \mu) \) for \( 1 < p < +\infty \), one of which is periodic, admits a dominated estimate with constant \( p/p - 1 \). In establishing this, the following analogue of Linderholm's theorem is obtained: Let \( \sigma \) and \( \varepsilon \) be two commuting non-singular point transformations of a Lebesgue space with \( \tau \) periodic. Then given \( \varepsilon > 0 \), there exists a periodic non-singular point transformation \( \sigma' \) such that \( \sigma' \) commutes with \( \tau \) and \( \mu(\{x : \sigma'x \neq \sigma x\}) < \varepsilon \). By an approximation argument, the principal result is applied to the convex combination of two isometries of \( L_p(0,1) \) induced by point transformations of the form \( \tau x = x^k, k > 0 \) to show that such convex combinations admit a dominated estimate with constant \( p/p - 1 \).

1. Introduction

In what follows we assume \( p \) fixed, \( 1 < p < +\infty \). Let \( (X, \mathcal{F}, \mu) \) be a \( \sigma \)-finite measure space, and let \( T \) be a linear operator mapping \( L_p(X, \mathcal{F}, \mu) \) into \( L_p(X, \mathcal{F}, \mu) \). If there exists a constant \( c > 0 \) such that

\[
\sup_n \left( \int |f|^p, \left| \frac{f + Tf}{2} \right|^p, \ldots, \left| \frac{f + \cdots + T^{n-1}f}{n} \right|^p, \ldots \right) d\mu \leq c^p \int |f|^p d\mu
\]

for each \( f \in L_p(X, \mathcal{F}, \mu) \), then we say that \( T \) admits a dominated estimate with constant \( c \). If \( \| Tf \|_p = \| f \|_p \) for each \( f \in L_p(X, \mathcal{F}, \mu) \), then we say that \( T \) is an isometry. If \( T \) maps non-negative functions to non-negative functions, then we say that \( T \) is positive.

Our main result is that a convex combination of two positive, invertible, commuting isometries, one of which is periodic, admits of a dominated estimate with constant \( p/p - 1 \). To establish this, we will prove an analogue of Lin-

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derholm's Theorem to show that if $\tau_1$ and $\tau_2$ are commuting non-singular point transformations with $\tau_2$ periodic (see Section 2 for definitions), then for every $\varepsilon > 0$, there exists a periodic non-singular point transformation $\tau_\varepsilon$ such that $\tau_\varepsilon$ commutes with $\tau_2$ and $\mu\{x: \tau_\varepsilon x \neq \tau_1 x\} < \varepsilon$. In Section 3, we apply the principal result to show that a convex combination of isometries of $L_p(0,1)$ of the form $Tf(x) = f(x^k) \cdot (kx^{k-1})^{1/p}$ admits of a dominated estimate with constant $p/p - 1$.

2. An analogue of Linderholm's theorem

In this section we will assume that $(X, \mathcal{F}, \mu)$ is a Lebesgue space, i.e., that it is separable, complete, non-atomic, and $\mu(X) = 1$. Let $\tau$ be a point transformation of $X$ into itself. If $\tau$ is one-to-one, measurable in the sense that $\tau A \in \mathcal{F}$ if and only if $A \in \mathcal{F}$, and if $\mu(\tau A) = 0$ if and only if $\mu(A) = 0$, we say that $\tau$ is non-singular. If there exists an integer $N$ such that for almost all $x \in X$ we have $\tau^N x = x$, we say that $\tau$ is periodic. If there exists an integer $n$ such that for almost all $x$ belonging to a set $A$ we have $\tau^n x = x$, where $n$ is the least such integer, we say that $\tau$ has period $n$ on $A$.

The main result of this section is the following:

**THEOREM 2.1.** Let $\tau$ and $\sigma$ be two non-singular point transformations of the Lebesgue space $(X, \mathcal{F}, \mu)$ with $\tau$ periodic. Then given $\varepsilon > 0$, there exists a periodic non-singular point transformation $\sigma^1$ of $(X, \mathcal{F}, \mu)$ such that $\sigma^1$ commutes with $\tau$ and

$$\mu\{x: \sigma^1 x \neq \sigma x\} < \varepsilon.$$ 

This is a generalization of Linderholm's approximation theorem:

**LINDERHOLM'S APPROXIMATION THEOREM.** Let $\sigma$ be a non-singular point transformation of the Lebesgue space $(X, \mathcal{F}, \mu)$ and let $\varepsilon > 0$. Then there exists a periodic point transformation $\tau$ such that

$$\mu\{x: \tau x \neq \sigma x\} < \varepsilon.$$ 

In [3], p. 71, there is a proof of this theorem in the measure preserving case that is easily adaptable to the non-singular case.

The bulk of the proof of Theorem 2.1 is contained in the following three lemmas.

**LEMMA 2.1.** Let $\tau$ and $\sigma$ be two commuting non-singular point transformations of the Lebesgue space $(X, \mathcal{F}, \mu)$ such that $\tau$ is periodic with period $n$ and $\sigma$ is anti-periodic. Then for every integer $m$, there exists a measurable set $A$ of positive measure such that the sets