PRIME FLOWS IN TOPOLOGICAL DYNAMICS

BY

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ABSTRACT

We present some results in topological dynamics and number theory. The number-theoretical results are estimates of the rates of convergence of sequences

\[ \left\{ \frac{1}{n} \sum_{i=0}^{n-1} \chi_{[0,\beta)}(i\alpha) : n > 0 \right\}, \]

where \( na \) is irrational, \( a \) is taken mod 1, and \( 0 < \beta < 1 \). One of these results is used to construct a homorphism \( T \) of a compact metric space \( X \) such that the minimal flow \( (X,T) \) had no nontrivial homomorphic images, i.e. is a prime flow. We define an infinite family of such flows, and describe other interesting properties of these flows.

Section 1

Certain familiar concepts in number theory have analogues in topological dynamics. In this paper, we shall be concerned with the notion of a minimal prime flow, i.e., a minimal flow with no factors except the trivial ones. Easy examples of such flows are the minimal flows on a prime number of points. Here, we shall construct an infinite prime flow. This example has a surprisingly easy realization; it is essentially obtained by carefully introducing a delay into an irrational rotation on the circle. Another realization in the bisequence space on \( \{0,1\} \) is obtained by doubling the ones in a Sturmian sequence. Thus, by starting with an equicontinuous flow with many obvious factors, we can construct a minimal, strictly ergodic flow on a metric space which has no factors.

In Section 1, we introduce the class of flows which will be the subject of our discussion, and indicate preliminary properties such as weak mixing and mini-

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mality. In Section 2, we prove a technical result which is used in Section 3. In Section 2, we also prove some related results about irregularities of distribution in an arbitrary minimal flow, and point out the implications of these results for some problems in diophantine approximation. In Section 3, we prove that our examples possess a property which we call POD, and in Section 4 we show several properties of POD flows, including their primitivity. For example, a POD flow is disjoint from every minimal flow except extensions of itself, and is disjoint from every power of itself.

We first define the flows which will concern us. A detailed proof of our assertions can be found in [6, Example 1]. By a flow $(X, T)$ we mean a homomorphism $T$ of a compact Hausdorff space. We let $K$ denote $[0,1)$ with addition modulo one. Let $\alpha \in K$ be irrational and $\beta \neq 0$. Define $f: K \to \{0,1\}$ by $f(y) = \chi_{(0,\beta)}(y)$ and for $n \in \mathbb{Z}$, define $x_0(n) = f(n\alpha)$. Then $x_0 \in S = \Pi_\alpha \{0,1\}$ and if $\sigma: S \to S$ is the shift on $S$ ($\sigma x(n) = x(n+1)$), and if $X = \sigma(x_0)^-$ (the orbit closure of $x$), then the flow $(X, \sigma)$ is called a Sturmian flow of type $(\alpha, \beta)$. Define $T_\beta: K \to K$ by $T_\beta(\gamma) = \gamma + \alpha$. We assume $\beta \notin \mathbb{Z}\alpha$.

**Proposition 1.1** (cf. [6, Th. 4.1]). The flows $(X, \sigma)$ and $(K, T_\beta)$ are minimal. There is a homomorphism $\rho: (X, \sigma) \to (K, T_\beta)$, $\rho(\sigma^nx_0) = nx_0$, such that $\rho^{-1}(\gamma)$ is a singleton unless $\gamma \in E = (\mathbb{Z}\alpha) \cup (\beta + \mathbb{Z}\alpha)$, in which case $\rho^{-1}(\gamma)$ is two points. Moreover, if $\rho^{-1}(0) = \{x_0, \bar{x}_0\}$ and $\rho^{-1}(\beta) = \{y_0, \bar{y}_0\}$ with $x_0(0) = y_0(0) = 1$, then $\bar{x}_0(0) = \bar{y}_0(0) = 0$ and for $n \neq 0$, $x_0(n) = \bar{x}_0(n)$ and $y_0(n) = \bar{y}_0(n)$.

Next we define the flow induced from $(X, \sigma)$ which will turn out to be a prime flow. Let $B = \{x \in X \mid x(0) = 1\}$, and let $A$ be a homeomorphic copy of $B$ such that $X \cap A = \emptyset$, with $\phi: B \to A$ a homeomorphism. Let $Y = X \cup A$ and define $\Psi: Y \to Y$ by

$$
\Psi(y) = \begin{cases} 
\phi(y) & \text{if } y \in B \\
\sigma(\phi^{-1}(y)) & \text{if } y \in A \\
\sigma(y) & \text{if } y \in X - B.
\end{cases}
$$

If we ignore most of the "doubled" points in $X$ by assuming that $B$ looks like $[0,\beta]$ and $X - B$ looks like $[\beta, 1]$, then a picture of $Y$ might be as shown in Fig. 1, where the action of $\sigma$ on $X$ is given by solid arrows and the action of $\Psi$ by dashed arrows. Note that $y = \sigma^2 x = \Psi^4 x$.

**Proposition 1.2** (cf. [6, remark following Th. 4.1]). $(Y, \Psi)$ is weakly mixing and minimal.