ON PELCZYNSKI'S PAPER
"UNIVERSAL BASES"

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ABSTRACT
A short proof is given to a theorem of Pelczynski concerning universal bases.

The main purpose of this paper is to give a simple proof to the following
theorem essentially due to A. Pelczynski [5].

THEOREM 1. The following two families of bases contain complementably
universal elements:

a) The family of all unconditional bases.
b) The family of all bases.

It is proved also that the family of all unconditional basic sequences in $L_p(0, 1)$,
$1 \leq p \leq \infty$, contains a complementably universal element.

Let us recall that a basis $(x_n)_{n=1}^\infty$ is called complementably universal for a
family $\mathcal{B}$ of bases provided that each basis in $\mathcal{B}$ is equivalent to a subbasis of
$(x_n)_{n=1}^\infty$, on which the canonical projection is bounded.

We refer the reader to [4] for notions which are not explained here.

PROOF OF THEOREM 1a. Let $(x_n)_{n=1}^\infty$ be a dense sequence in $C(0, 1)$. Define a
norm $\| \cdot \|$ on the linear space $L$ of all eventually zero sequences of scalars by:

$$\| (a_n) \| = \sup \left\{ \| \sum_{n=1}^N \varepsilon_n \cdot a_n \cdot x_n \| : N = 1, 2, \ldots, | \varepsilon_n | = 1, n = 1, 2, \ldots, N \right\}.$$ (1)

Denote by $\bar{x}_n$ the $n$th unit vector in $L$; then it is clear that $(\bar{x}_n)_{n=1}^\infty$ constitutes
an unconditional basis for the completion of $L$ by the norm $\| \cdot \|$. If $(y_n)_{n=1}^\infty$ is

* This is part of the author's Ph.D. thesis written at the Hebrew University of Jerusalem under
the supervision of Professor J. Lindenstrauss. The author wishes to thank Professor Lindenstrauss
for his guidance.

Received October 15, 1975
an unconditional basis for a (separable) Banach space then by the Banach-Mazur theorem we can assume without loss of generality that $(y_n)^{n-1} \subset C(0, 1)$. By [1] we can choose a subsequence $(x_n)^{n-1}$ of $(x_n)^{n-1}$ which is equivalent to $(y_n)^{n-1}$. In particular $(x_n)^{n-1}$ is an unconditional basic sequence and thus equivalent to $(x_n)^{n-1}$. By the unconditionality of $(x_n)^{n-1}$ the canonical projection onto $(x_n)^{n-1}$ is bounded.

**Proof of Theorem 1b.** Let $(x_n)^{n-1}$ be as in the proof of the first part. For all $k$-tuples of positive integers $i_1, i_2, \cdots, i_k$ $(k = 1, 2, \cdots)$ define

$$y(i_1, \cdots, i_k) = x_{i_k}.$$  

Let $\varphi$ be a one to one function from the set of all finite sequences of positive integers onto the set of positive integers with the property that for all $i_1, i_2, \cdots, i_k$:

$$\varphi(i_1, \cdots, i_{k-1}) < \varphi(i_1, \cdots, i_k).$$

Put $\mathcal{A} = \{((i_1), (i_1, i_2), \cdots, (i_1, \cdots, i_k)); k, i_1, \cdots, i_k = 1, 2, \cdots\}$ and define a norm $\| \cdot \|$ on $L$ by

$$\| (a_n) \| = \sup \left\{ \sum_{n=1}^{\infty} \chi_A(\varphi^{-1}(n)) \cdot a_n \cdot y(\varphi^{-1}(n)); N = 1, 2, \cdots; A \in \mathcal{A} \right\}$$

($\chi_A$ denotes the characteristic function of $A$).

It is easily checked that $(\tilde{x}_n)^n_{n-1}$, the unit vectors of $L$, form a basis for the completion under $\| \cdot \|$ of $L$.

If $(i_n)^n_{n-1}$ is a sequence of positive integers then (4) and the fact that for all $A \in \mathcal{A}$ and all $B \in \mathcal{A}$ we have $A \cap B \in \mathcal{A}$ imply that the canonical projection onto $[\tilde{x}_{\varphi(i_1, \cdots, i_n)}]^n_{n-1}$ is bounded.

If $(z_n)$ is a basis of a Banach space then as in the proof of the first part there exists a subsequence $(x_n)^n_{n-1}$ of $(x_n)^n_{n-1}$ which is equivalent to $(z_n)^n_{n-1}$. In particular $(x_n)^n_{n-1} = (y(i_1, i_2, \cdots, i_n))^n_{n-1}$ is a basic sequence and thus by (3), (4) and the fact that

$$\{((i_1), (i_1, i_2), \cdots, (i_1, \cdots, i_k)) \cap A \mid A \in \mathcal{A}\}$$

$$= \{((i_1), (i_1, i_2), \cdots, (i_1, \cdots, i_l)); l = 1, 2, \cdots, k\} \cup \emptyset$$

we get that for all scalars $(a(i_1, \cdots, i_n))^n_{n-1}$: