SETS WITH A MODE

BY

GEORGE CONVERSE

ABSTRACT

Let \( M \) be a point and \( S \) be a compact set in \( \mathbb{R}^2 \) such that \( S \) is the closure of its interior. The theorem desired says that if \( M \) is a mode of \( S \) then \( S \) is convex and centrally symmetric with respect to \( M \). Some conditions on the boundary of \( S \) are needed for the proof given.

Throughout this paper \( S \) will be a nonempty compact subset of \( \mathbb{R}^2 \) which is the closure of its interior and \( M \) will be a mode of \( S \) (defined below). In their paper Dharmadhikari and Jogdeo [1] prove that \( S \) is convex and hence centrally symmetric with respect to \( M \) provided \( S \) has Jordan polygonal boundary. The aim of this paper is to replace the condition of a Jordan polygonal boundary with a condition satisfied by all compact convex sets. The condition for this paper is that the boundary of \( S \) will consist of a finite number of acceptable closed curves (defined below) which meet in at most a finite number of points.

DEFINITIONS. For any real number \( t \) and any unit vector \( u \) in \( \mathbb{R}^2 \), let \( L(u, t) \) be the line

\[ \{ z \in \mathbb{R}^2 : \langle u, z - M \rangle = t \} \]

and \( m \) be Lebesgue measure on the line. \( M \) is a mode of \( S \) if, for each \( u \), \( m(L(u, t) \cap S) \) is a nonincreasing function of \( t \) for \( t \geq 0 \) and a nondecreasing function of \( t \) for \( t \leq 0 \).

A curve is an acceptable closed curve if there is a homeomorphism \( f \) of the unit circle ([0, 2\( \pi \]) with 0 and 2\( \pi \) identified) onto the curve such that \( f \) has a nonzero left derivative \( f'_L \) everywhere on \((0, 2\pi]\), \( f \) has a nonzero right derivative \( f'_R \) everywhere on \([0, 2\pi)\), \( f'_L \) is continuous from the left, \( f'_R \) is continuous from the right and \( f'_L = f'_R = f' \) except for at most a countable number of points.

The purpose of this paper is to prove the following

Received December 15, 1975 and in revised form August 26, 1976
THEOREM. Let $S$ be a nonempty compact subset of $\mathbb{R}^2$ which is the closure of its interior. Suppose $S$ has a mode $M$ and the boundary of $S$ consists of a finite number of acceptable closed curves which meet in at most a finite number of points. Then $S$ is convex and centrally symmetric with respect to $M$.

Before proceeding with the proof, there are a few consequences of the condition on the boundary of $S$ that should be noted to give a more geometric idea of what an acceptable closed curve is and how the condition will be used. First the existence of nonzero left and right derivatives at a point imply the existence of tangent rays at that point. If $f' = f'$ the tangent rays are the two opposite rays of the tangent line.

Second, if $f$ is the homeomorphism guaranteed by the definition, then

$$f(b) - f(a) = \int_a^b f'(x)dx, \quad 0 \leq a < b \leq 2\pi.$$ 

This may be concluded from exercise 18:41d in Hewitt and Stromberg [2] or from 8:11 (or the proof of 8:21) in Rudin [6].

Third, the image of $f$ has finite length given by, for example, $\int_0^{2\pi} |f'(x)| dx$ [3, p. 36]. Thus the boundary of $S$ has finite length. No precise definition of length will be needed but elementary calculus such as Purcell 16:4 [4] will be used as will the fact that if one side of a rectangle intersects $S$ in a length $l$ greater than the opposite side then the length of the boundary of $S$ inside the rectangle is at least $l$.

Finally, it follows from general knowledge (or [5, 24.1]) that the boundary of a compact convex set with nonempty interior is an acceptable closed curve. The proof of the Theorem now follows with some notation and a sequence of lemmas.

NOTATION. For any angle $t$, let

$$R(t) = \{x \in \mathbb{R}^2 : x = M + a(\cos t, \sin t), a > 0\}.$$

If $x$ in the boundary of $S$ has a tangent line denote it by $T(x)$. If $x \neq y$, let $(x, y)$ be the open line segment between $x$ and $y$. If an endpoint is to be included, a square bracket will replace the appropriate parenthesis. Let $\partial S$ denote the boundary of $S$ and $d$ be Euclidean distance in $\mathbb{R}^2$.

LEMMA 1. $m\{t \in [0, 2\pi] : R(t) \subset T(x) \text{ for some } x \in \partial S\} = 0$.

PROOF. For any $t$ in $(0, 2\pi)$, let $s(t)$ be the length of the boundary of $S$ within the angle from $R(0)$ to $R(t)$. Then $s$ is a monotone increasing function so