SETS WITH A MODE

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ABSTRACT
Let $M$ be a point and $S$ be a compact set in $\mathbb{R}^2$ such that $S$ is the closure of its interior. The theorem desired says that if $M$ is a mode of $S$ then $S$ is convex and centrally symmetric with respect to $M$. Some conditions on the boundary of $S$ are needed for the proof given.

Throughout this paper $S$ will be a nonempty compact subset of $\mathbb{R}^2$ which is the closure of its interior and $M$ will be a mode of $S$ (defined below). In their paper Dharmadhikari and Jogdeo [1] prove that $S$ is convex and hence centrally symmetric with respect to $M$ provided $S$ has Jordan polygonal boundary. The aim of this paper is to replace the condition of a Jordan polygonal boundary with a condition satisfied by all compact convex sets. The condition for this paper is that the boundary of $S$ will consist of a finite number of acceptable closed curves (defined below) which meet in at most a finite number of points.

DEFINITIONS. For any real number $t$ and any unit vector $u$ in $\mathbb{R}^2$, let $L(u, t)$ be the line

$$\{z \in \mathbb{R}^2 : \langle u, z - M \rangle = t\}$$

and $m$ be Lebesgue measure on the line. $M$ is a mode of $S$ if, for each $u$, $m(L(u, t) \cap S)$ is a nonincreasing function of $t$ for $t \geq 0$ and a nondecreasing function of $t$ for $t \leq 0$.

A curve is an acceptable closed curve if there is a homeomorphism $f$ of the unit circle ($[0, 2\pi]$ with $0$ and $2\pi$ identified) onto the curve such that $f$ has a nonzero left derivative $f'_L$ everywhere on $(0, 2\pi]$, $f$ has a nonzero right derivative $f'_R$ everywhere on $[0, 2\pi)$, $f'_L$ is continuous from the left, $f'_R$ is continuous from the right and $f'_L = f'_R = f'$ except for at most a countable number of points.

The purpose of this paper is to prove the following

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THEOREM. Let \( S \) be a nonempty compact subset of \( \mathbb{R}^2 \) which is the closure of its interior. Suppose \( S \) has a mode \( M \) and the boundary of \( S \) consists of a finite number of acceptable closed curves which meet in at most a finite number of points. Then \( S \) is convex and centrally symmetric with respect to \( M \).

Before proceeding with the proof, there are a few consequences of the condition on the boundary of \( S \) that should be noted to give a more geometric idea of what an acceptable closed curve is and how the condition will be used. First the existence of nonzero left and right derivatives at a point imply the existence of tangent rays at that point. If \( f' = f' \) the tangent rays are the two opposite rays of the tangent line.

Second, if \( f \) is the homeomorphism guaranteed by the definition, then
\[
f(b) - f(a) = \int_a^b f'(x)dx, \quad 0 \leq a < b \leq 2\pi.
\]
This may be concluded from exercise 18:41d in Hewitt and Stromberg [2] or from 8:11 (or the proof of 8:21) in Rudin [6].

Third, the image of \( f \) has finite length given by, for example, \( \int_0^{2\pi} |f'(x)| dx \) [3, p. 36]. Thus the boundary of \( S \) has finite length. No precise definition of length will be needed but elementary calculus such as Purcell 16:4 [4] will be used as will the fact that if one side of a rectangle intersects \( S \) in a length \( l \) greater than the opposite side then the length of the boundary of \( S \) inside the rectangle is at least \( l \).

Finally, it follows from general knowledge (or [5, 24.1]) that the boundary of a compact convex set with nonempty interior is an acceptable closed curve. The proof of the Theorem now follows with some notation and a sequence of lemmas.

NOTATION. For any angle \( t \), let
\[
R(t) = \{x \in \mathbb{R}^2 : x = M + a(\cos t, \sin t), a > 0\}.
\]
If \( x \) in the boundary of \( S \) has a tangent line denote it by \( T(x) \). If \( x \neq y \), let \((x, y)\) be the open line segment between \( x \) and \( y \). If an endpoint is to be included, a square bracket will replace the appropriate parenthesis. Let \( \partial S \) denote the boundary of \( S \) and \( d \) be Euclidean distance in \( \mathbb{R}^2 \).

LEMMA 1. \( m \{t \in [0, 2\pi] : R(t) \cap T(x) \text{ for some } x \in \partial S\} = 0 \).

PROOF. For any \( t \) in \((0, 2\pi)\), let \( s(t) \) be the length of the boundary of \( S \) within the angle from \( R(0) \) to \( R(t) \). Then \( s \) is a monotone increasing function so