HOW OFTEN IS A POLYGON BOUNDED BY THREE SIDES?

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ABSTRACT
Let \( L_n \) be the set of lines (no two parallel) determining an \( n \)-sided bounded face \( F \) in the Euclidean plane. We show that the number, \( f(L_n) \), of triples from \( L_n \) that determine a triangle containing \( F \) satisfies

\[
n - 2 \leq f(L_n) \leq \frac{n}{6} \left[ \frac{n^2 - 1}{4} \right]
\]

and these bounds are best. This result is generalized to \( d \)-dimensional Euclidean space (without the claim that the upper bound is attainable).

1. Introduction and notation

Let \( L_n \) be a set of \( n \geq 3 \) lines in general position in the Euclidean plane such that some bounded face \( F \) determined by \( L_n \) is \( n \)-sided. We let \( f(L_n) \) be the number of triples of lines of \( L_n \) that form a triangle containing \( F \) and will prove the following result in the third section:

**Theorem 1.**

\[
n - 2 \leq f(L_n) \leq \frac{n}{6} \left[ \frac{n^2 - 1}{4} \right]
\]

and these bounds are best.

Our proof of this result will involve the consideration of certain properties of a set \( P \) of points \( p_i, 1 \leq i \leq m \), in general position on a circle \( S \) (no two antipodal). Each pair of distinct points \( p_i \) and \( p_j \) determine two arcs of \( S \): the “arc \( p_ip_j \)” will mean the smaller of these and we refer to the set of all such smaller arcs as “the arcs of \( P \”). A triple of points of \( P \) not contained in any semicircle of \( S \) will be called a central triple (since the triangle formed by these points contains the center of \( S \)). The point of \( S \) antipodal to \( p_i \) will be denoted by \( p'_i, 1 \leq i \leq n \). We note that a triple \( \{p_i, p_j, p_k\} \) is central iff the arc \( p_ip_k \) contains \( p'_i \).

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2. Some properties of points on a circle

**Lemma 1.** Let $P$ be a set of points $p_i, 1 \leq i \leq n$, in general position on a circle $S$.

(i) The union of the arcs of $P$ is $S$ or is contained in a semicircle of $S$.

(ii) If one point of $P$ is a member of a central triple then each point is.

(iii) If every three points of $P$ are contained in a semicircle of $S$ then they all are.

**Proof.**

(i) If the union of the arcs of $P$ is not $S$ it is an arc $C$ with endpoints $p_i$ and $p_j$, say. The arc $p_ip_j$ is either $S-C$ or $C$; in either case we are done.

(ii) If some point $p_i$ is not a member of any central triple then the antipodal point $p'_i$ cannot lie on any arc of $P$ and so by (i) all points of $P$ lie in a semicircle. But then no point of $P$ is a member of a central triple.

(iii) The point $p_i$ is not a member of a central triple so that $p'_i$ does not lie in any arc of $P$. By (i) some semicircle of $S$ contains all points of $P$.

**Lemma 2.** Let $P_n$ be a set of points $p_i, 1 \leq i \leq n$, $n \geq 3$, in general position on a circle $S$ and let $g(P_n)$ be the number of central triples of members of $P_n$. Then, if $g(P_n) > 0$, we have

$$n - 2 \leq g(P_n) \leq \frac{n}{6} \left[ \frac{n^2 - 1}{4} \right]$$

and these bounds are best.

**Proof.** Let $x_i$ and $y_i$ be the numbers of points $p_i \neq p_j$ in the two semicircles of $S$ with endpoint $p_i$. Then $x_i + y_i = n - 1$ and the number of triples not central is

$$\left( \begin{array}{c} n \cr 3 \end{array} \right) - g(P_n) = \frac{1}{2} \sum_{i=1}^{n} \left[ \left( \begin{array}{c} x_i \cr 2 \end{array} \right) + \left( \begin{array}{c} y_i \cr 2 \end{array} \right) \right] = -\frac{1}{2} \left( \begin{array}{c} n \cr 2 \end{array} \right) + \frac{1}{4} \sum_{i=1}^{n} (x_i^2 + y_i^2)$$

so that

$$g(P_n) \leq \left( \begin{array}{c} n \cr 3 \end{array} \right) + \frac{1}{2} \left( \begin{array}{c} n \cr 2 \end{array} \right) - \frac{1}{4} \left[ \begin{array}{l} 2n \left( \frac{n-1}{2} \right)^2, \text{ n odd} \\
 \left( \begin{array}{c} n \cr 4 \end{array} \right) (\frac{n^2}{4} + \frac{n-2}{4})^2, \text{ n even} \end{array} \right]$$

so that

$$g(P_n) \leq \left\{ \begin{array}{l} \frac{n(n^2-1)}{24}, \text{ n odd} \\
 \frac{n(n^2-4)}{24}, \text{ n even} \end{array} \right\} = \frac{n}{6} \left[ \frac{n^2 - 1}{4} \right].$$