THE GROUP ALGEBRA OF THE
INFINITE SYMMETRIC GROUP

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ABSTRACT

The rational group algebra of the infinite symmetric group is studied using Young diagrams. Maximal and prime ideals are characterized and the maximal condition on ideals is proved.

Let $S_n$ denote the symmetric group of degree $n$, the group of permutations of $\{1, 2, \ldots, n\}$. There are inclusions

$$S_1 \subset S_2 \subset S_3 \cdots$$

and $S$ denotes the union of the $S_n$. $S$ can also be described as the group of those permutations of a countable set which move only finitely many points.

The purpose of this paper is to investigate the group algebra $F[S]$, where $F$ is a field of characteristic zero, particularly its ideal structure. $F[S]$ is the ascending union of the group algebras $F[S_n]$, and the ideal structure of each $F[S_n]$ is given by the theory of Young diagrams. The set of Young diagrams is a partially ordered set and there is a one-to-one correspondence between ideals of $F[S]$ and certain collections of Young diagrams. By studying the Young diagrams, conclusions can be drawn about $F[S]$.

One reason for studying $F[S]$ is that, because of the Young theory, it can be investigated far more thoroughly than the group algebra of an arbitrary locally finite group and so may give insights which have wider application. However, we feel that the main interest of our work is that $F[S]$ turns out to have a number of curious properties: $F[S]$ satisfies the ascending chain condition on ideals—in fact, every ideal is principal; any sum of prime ideals of $F[S]$ is a prime ideal or all of $F[S]$; and $F[S]$ has precisely two maximal ideals. We do not know

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whether these properties are shared by a large class of locally finite group algebras or whether they are just accidental properties of $F[S]$.

1. Young diagrams

In this section we describe the relation between Young diagrams and ideals of $F[S]$. For our purposes all that matters is the shape of the Young diagram, which means that we are really only talking about partitions of $n$. However, we will use the Young diagram terminology since it makes certain "geometric" observations easier and follows common usage. We will only use the most basic facts of the theory.

Let $n$ be a positive integer. With each partition
\[
\{n_1 \geq n_2 \cdots \geq n_k > 0: n_1 + \cdots + n_k = n\}
\]
of $n$, the associated Young diagram is the planar array (part of a chessboard) of $k$ rows with $n_i$ boxes in the $i$th row. For example, the partition $\{5, 2, 2, 1\}$ corresponds to the Young diagram

\[
\begin{array}{cccc}
\hline
\ & \ & \ & \\
\ & \ & \ & \\
\ & \ & \ & \\
\ & \ & \ & \\
\hline
\end{array}
\]

The set $\mathcal{D}$ of all Young diagrams (of all sizes) is partially ordered as follows: If $A = \{a_1, \cdots, a_r\}$ and $B = \{b_1, \cdots, b_s\}$ are Young diagrams, then $A \geq B$ if $r \geq s$ and $a_i \geq b_j$ for $i = 1, \cdots, s$. In other words, as planar diagrams $A$ is obtained from $B$ by adjoining boxes. The join of $A$ and $B$ is

\[
A \vee B = \{\max(a_i, b_j), \max(a_2, b_2), \cdots\}
\]

where, by convention, $a_i = 0$ if $i > r$, $b_j = 0$ if $j > s$. $A \vee B$ is the smallest Young diagram greater than both $A$ and $B$.

The fundamental theorem which follows gives the relation between these diagrams and the ideal structure of $F[S]$. Recall that "ideal" always means "two-sided ideal".

Theorem 1. (See [1, 4.27, 4.51, 4.52].)

1) Let $D_1, \cdots, D_t$ be the Young diagrams of size $n$. Then $F[S_n]$ has $t$ irreducible orthogonal central idempotents $e(D_1), \cdots, e(D_t)$ and each simple factor $e(D_i)F[S_n]$ is isomorphic to a full matrix ring over $F$. 