LOOSE BLOCK INDEPENDENCE

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ABSTRACT
A finite state stationary process is defined to be loosely block independent if long blocks are almost independent in the \( \bar{f} \) sense. We show that loose block independence is preserved under Kakutani equivalence and \( \bar{f} \) limits. We show directly that any loosely block independent process is the \( \bar{f} \) limit of Bernoulli processes and is a factor of a process which is Kakutani equivalent to a Bernoulli shift. The existing equivalence theory then yields that the loosely block independent processes are exactly the loosely Bernoulli (or finitely fixed) processes.

1. Introduction

In their studies of Kakutani equivalence, Jack Feldman and Anatole Katok independently introduced a monotone matching metric, now called \( \bar{f} \). They use it to define loosely Bernoulli, a notion analogous to very weak Bernoulli with the new metric replacing Ornstein's \( \bar{d} \). The loosely Bernoulli property is seen to play a fundamental role in the equivalence theory since a process is loosely Bernoulli if and only if it is Kakutani equivalent either to a Bernoulli shift (independent identically distributed process) or to a rotation of the circle \([1, 3, 6]\).

In this article we introduce another \( \bar{f} \) concept called loose block independence, analogous to Paul Shield's \( \bar{d} \) notion of almost block independence. This property is stable under building towers, inducing on sets, taking factors, and taking \( \bar{f} \)-limits. The collection of loosely block independent processes is easily seen to contain all Bernoulli shifts and circle rotations and hence must contain their entire Kakutani equivalence classes.

Moreover, the Bernoulli processes are \( \bar{f} \)-dense in the class of loosely block independent processes; and every loosely block independent process is a factor of a transformation which is Kakutani equivalent to a Bernoulli shift. The standard equivalence theory \([6]\) yields immediately that the loosely block
independent processes are exactly those which are Kakutani equivalent to Bernoulli shifts or to rotations.

2. Definitions

Let $A$ be a finite index set. Throughout this article $T$ will be an invertible measure preserving transformation on a probability space $(X, \mu)$; and $P = \{P_a : a \in A\}$ will be a measurable partition of $X$. The process $(T, P)$ determines a distribution measure (which we shall also denote by $\mu$) on $A^\mathbb{Z}$, the space of all doubly infinite sequences from $A$. For $n$ a positive integer, the projection of $\mu$ onto $A^n$ will be denoted by $\mu_n$.

We recall the following definitions. For $B \subseteq X$, $\mu(B) > 0$, and $x \in B$, the return time is $n(x) = \inf\{k > 0 : T^k x \in B\}$. This function is finite a.e. and defines the induced transformation $T_n$ by $T_n(x) = T^{n(x)} x$. The notion dual to that of an induced transformation is that of a tower transformation. If $h$ is an integrable function on $X$ with values in the set of positive integers, we define the tower space $X^h$ by $X^h = \{(x, i) : 1 \leq i \leq h(x)\}$, with normalized measure inherited from that on $X$. The tower transformation $T^h$ is defined by $T^h(x, i) = (x, i + 1)$ for $1 \leq i < h(x)$ and $T^h(x, h(x)) = (Tx, 1)$. A partition $P$ on $X$ extends to a standard partition $P^h$ on $X^h$ consisting of the complement of the base of the tower adjoining the collection of sets in $P$.

Transformations $T$ and $S$ on spaces $(X, \mu)$ and $(Y, \nu)$, respectively, are said to be Kakutani equivalent if there exist sets $B \subseteq X$, $\mu(B) > 0$ and $C \subseteq Y$, $\nu(C) > 0$ with $T_B$ isomorphic to $S_C$. Equivalently, in the dual formulation $T$ and $S$ are Kakutani equivalent if there exist tower functions $h$ and $k$ with $T^h$ isomorphic to $S^k$.

In this paragraph we sketch the essentials of the $\bar{f}$-metric. Let $A$ be a finite index set and let $n$ be a positive integer. For $x, y \in A^n$ we define $\bar{f}(x, y) = 1 - k/n$, where $k$ is the largest integer for which there exist sequences $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$ such that $x(i_l) = y(j_l)$ for $1 \leq l \leq k$. For processes $(T, P)$ and $(S, Q)$ indexed by the same set $A$, with corresponding distributions $\mu$ and $\nu$ on $A^\mathbb{Z}$, we set $\bar{f}_n((T, P), (S, Q)) = \inf\{\epsilon : \text{there is a measure } \rho \text{ on } A^n \times A^n \text{ with marginals } \mu_n \text{ and } \nu_n \text{ such that } \rho\{(x, y) : \bar{f}_n(x, y) < \epsilon\} > 1 - \epsilon\}$. We further define $\bar{f}((T, P), (S, Q)) = \limsup \bar{f}_n((T, P), (S, Q))$.

Given a process $(T, P)$, the independent $N$-blocking, $(T, P)_N$, of $(T, P)$ is the (usually non-stationary) process associated with the product measure $(\mu_N)^\mathbb{Z}$.

**Definition.** A stationary process $(T, P)$ is loosely block independent (LBI) if for every $\epsilon > 0$ there exists an integer $M$ so that $\bar{f}((T, P), (T, P)_N) < \epsilon$ for all $N \geq M$. 