ON THE NUMBER OF SUBGRAPHS
OF PRESCRIBED TYPE OF GRAPHS
WITH A GIVEN NUMBER OF EDGES

BY
NOGA ALON

ABSTRACT
All graphs considered are finite, undirected, with no loops, no multiple edges
and no isolated vertices. For a graph $H = (V(H), E(H))$ and for $S \subseteq V(H)$
define $N(S) = \{x \in V(H) : xy \in E(H) \text{ for some } y \in S\}$. Define also $\delta(H) = \max\{|S| - |N(S)| : S \subseteq V(H)\}$. For two graphs $G, H$
let $N(G, H)$ denote the number of subgraphs of $G$ isomorphic to $H$. Define also
for $l > 0$, $N(l, H) = \max N(G, H)$, where the maximum is taken over all graphs
$G$ with $l$ edges. We investigate the asymptotic behaviour of $N(l, H)$ for fixed $H$
as $l$ tends to infinity. The main results are:

THEOREM A. For every graph $H$ there are positive constants $c_1, c_2$ such that

$$c_1 \gamma^{\gamma(H)} \leq N(l, H) \leq c_2 \gamma^{\gamma(H)}$$

for all $l \geq |E(H)|$.

THEOREM B. If $\delta(H) = 0$ then

$$N(l, H) = (1 + O(l^{-1/2})) \cdot \frac{1}{|\text{Aut } H|} \cdot (2l)^{\gamma(H)/2},$$

where $|\text{Aut } H|$ is the number of automorphisms of $H$.

(It turns out that $\delta(H) = 0$ iff $H$ has a spanning subgraph which is a disjoint
union of cycles and isolated edges.)

Notations and definitions

All graphs considered in this paper are finite and simple (no loops, no multiple
edges) and have no isolated vertices. For every set $A$, $|A|$ is the cardinality of $A$.$G_l$ is a graph with $l$ edges. $K(n)$ is the complete graph on $n$ vertices ($n \geq 2$). $P(r)$
is the path of length $r$ ($r \geq 1$). $C(h)$ is the cycle of length $h$ ($h \geq 3$). $I(k)$ is the
graph consisting of $k$ independent edges (= disjoint union of $k$ $P(1)$'s), $k \geq 1$.$K(1, k)$ is the star consisting of $k$ edges incident with one common vertex.

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For every graph $G$: $V(G)$ is the set of vertices of $G$, $E(G)$ is its set of edges. $k(G) = \frac{1}{2}|V(G)|$. Aut $G$ is the group of automorphisms of $G$. $N(x)$ is the set of vertices adjacent to a vertex $x \in V(G)$. For $S \subseteq V(G)$ we put $N(S) = \bigcup \{N(x): x \in S\}$. If needed, we indicate the underlying graph by a subscript and write $N_G(S)$.

For every graph $G$ on $v$ vertices and every spanning subgraph $H$ of $G$, $x(G, H)$ is the number of subgraphs of $K(v)$, isomorphic to $G$, that contain a fixed copy of $H$ in $K(v)$. For every two graphs $G, H$, $N(G, H)$ is the number of subgraphs of $G$ isomorphic to $H$. For every graph $H$ and every positive integer $l$, $N(l, H)$ is the number of subgraphs of $G$ isomorphic to $H$. For every graph $H$ and every positive integer $l$, $N(l, H)$ is known for every complete graph $H$ and every positive integer $l$.

P. Erdős (private communication) posed the problem of determining or estimating $N(l, H)$ for other graphs. We shall investigate the asymptotic behaviour of $N(l, H)$ for fixed $H$ when $l$ tends to infinity.

By a theorem of Erdős and Hanani [2], or by a special case of the Kruskal-Katona Theorem (a simple proof of which is given in [1]) we know that if $l = (\frac{1}{2}) + r$, $0 \leq r \leq t$, then for every $v \geq 2$

$$N(l, K(v)) = \binom{l}{v} + \binom{r}{v-1}.$$  

It is also easy to check that

$$N(l, K(1, k)) = \binom{l}{k} \quad \text{and} \quad N(l, I(k)) = \binom{l}{k}.$$  

**Remark 1.** Obviously, for every graph $H$ with $k$ edges and for every $l$, $N(l, H) \leq \binom{k}{l}$. In this sense $K(1, k)$ and $I(k)$ are extremal, and we can prove that these are the only extremal graphs. As a matter of fact we can prove the following stronger result:

Let $H$ be a graph with $k$ edges and suppose that there exists an integer $l \geq k + 2$ such that $N(l, H) = \binom{l}{k}$. Then $H$ is isomorphic to either $K(1, k)$ or $I(k)$.

We can also show that if $|E(H)| = k$, then $N(k + 1, H) = k + 1 = \binom{k + 1}{k}$ iff $H$ is obtained from an edge-transitive graph $G$ by deleting an edge.

We shall not give the proofs as they are rather lengthy and not very complicated.

Our first theorem describes the asymptotic behaviour of $N(l, H)$ for any graph $H$ containing a perfect matching.

For any graph $H$ with a perfect matching, let $x(H) = x(H, I(k(H)))$ denote