BIFURCATION IN THE NEIGHBOURHOOD OF A NON-ISOLATED SINGULAR POINT

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ABSTRACT
The Lyapunov-Schmidt method for bifurcation problems has, until recently, been applied only to operator equations whose singular points are isolated in the solution set of the equation. For bifurcation at a multiple eigenvalue involving several parameters, however, singular points are often non-isolated. In this paper, the case of intersecting curves of singular points is considered. Under natural hypotheses on these curves, and assuming suitable transversality conditions on the first order nonlinearity of the operator, it is shown that the solution set of the equation may be completely determined locally in terms of the solutions of associated finite dimensional polynomial equations.

1. Introduction

Let $E$ and $Y$ be real Banach spaces. In this paper, we discuss the nature of the set of small solutions of a class of operator equations of the form

$$(1.1) \quad G(u) = 0, \quad u \in E$$

where $G: E \rightarrow Y$ is a mapping of class $C^n$ for some $n \geq 3$, $G(0) = 0$ and $DG(0): E \rightarrow Y$ is a Fredholm operator with index $m \geq 2$.

Since $DG(u)$ is assumed continuous in $u$, there exists a neighbourhood $U$ of zero in $E$ such that $DG(u)$ is a Fredholm operator of index $m$ for all $u \in U$ [6]. Suppose $u \in U$ is a solution of (1.1) and $DG(u)$ is onto $Y$. Then, by the implicit function theorem, the solution set of (1.1) is locally (i.e. near $u$) $C^n$ homeomorphic to an open ball in $\mathbb{R}^n$.

From the point of view of bifurcation theory, it is important to investigate the nature of the solution set of equation (1.1) in the neighbourhood of a point $u \in U$ such that $G(u) = 0$ and $DG(u)$ is not onto $Y$. Such points are called singular. Henceforth we shall suppose $u = 0$ is a singular point. Note that $E$ must now be at least three-dimensional.

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The set of small solutions of (1.1) may be visualised as consisting of an $m$-dimensional $C^r$ manifold, on which $DG(u)$ has range $Y$, together with a set $S$ of singular points. The simplest possible case, when $S \cap W = \{0\}$ for some neighbourhood $W$ of zero in $E$, has been discussed thoroughly by Magnus [9–11]. In this case, the singular point $u = 0$ is called isolated.

Magnus' analysis is of particular interest in bifurcation problems involving one parameter. Such problems may be formulated as an equation of the form (1.1) with $m = 1$ ([9], [14]).

Now, the range of $DG(0)$ is a subspace of $Y$ with codimension $\beta \geq 1$. If $\beta = 1$ and $m \geq 2$ then the singular point $u = 0$ cannot be expected to be isolated in general. If $\beta \geq 2$ and $m \geq 2$ then the case when $u = 0$ is an isolated singular point must be regarded as exceptional. Since bifurcation problems involving several parameters correspond to the case $m \geq 2$, it is important to consider classes of equation (1.1) such that $u = 0$ is a non-isolated singular point.

To further motivate this last statement, consider the usual formulation of bifurcation problems involving several parameters. Let $X$ be a real Banach space and let $E = \mathbb{R}^{p+1} \times X$ where $p \geq 1$. Write $G(u)$ as $G(\lambda, \mu, x)$ for $u = (\lambda, \mu, x) \in \mathbb{R} \times \mathbb{R}^p \times X$. Suppose $G(\lambda, 0, \theta) = 0$ for each $\lambda \in \mathbb{R}$, where $0 \in \mathbb{R}^p$, $\theta \in X$ are the zeroes of $\mathbb{R}^p, X$ respectively. $G_\lambda(0, 0, \theta) : X \to Y$ is assumed to be a Fredholm operator with index zero, null space $N \neq \{0\}$ and range $R$.

It is common to assume, for $p = 1$, that $G_\lambda(0, 0, \theta) \not\in R$ (e.g. [3, 4, 7, 8, 13]). In this case, $DG(0) : E \to Y$ has index two (i.e. $m = 2$) so that it is reasonable to assume that the singular point $u = 0 ((\lambda, \mu, x) = (0, 0, \theta))$ is isolated, provided that $N$ has dimension one or two. In fact, the knowledge of a line $\{(\lambda, 0, \theta) : \lambda \in \mathbb{R}\}$ of trivial solutions considerably helps the analysis, and Magnus' transversality conditions may be modified in order to retain and exploit the significance of the parameter space, as in [4]. However, when $\dim N \geq 3$ the assumption of an isolated singular point at zero is no longer acceptable in a theory attempting any degree of generality. It is worth noting that if $N$ has dimension one, then $DG(0, 0, \theta)$ is onto $Y$.

Another situation of interest arises when $p = 1$ and $G(\lambda, \mu, \theta) = 0$ for all $(\lambda, \mu) \in \mathbb{R}^2$. In this case, $G_\mu(0, 0, \theta) = 0$ so that $DG(0) : E \to Y$ again has index two ($m = 2$). If $N$ has dimension one ($\beta = 1$), it is conceivable that the singular point $\lambda = \mu = 0, x = \theta$ is isolated. However, this will only be the case if $\lambda G_\mu(0, 0, \theta) + \mu G_\lambda(0, 0, \theta)$ maps $N$ into $R$ for all $(\lambda, \mu) \in \mathbb{R}^2$; otherwise there exists a unique curve of singular points in $\mathbb{R}^2 \times \theta$ through zero in $E$. This fact follows easily from the analysis of Crandall and Rabinowitz [5] for bifurcation from simple eigenvalues (see [15] for details).