A WHITEHEAD THEOREM FOR LONG TOWERS OF SPACES

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ABSTRACT

We show that one can construct the universal $R$-homology isomorphism $K \to E_\infty X$ of Bousfield [1] by a transfinite iteration of an elementary homology correction map. This correction map is essentially the same as the one used classically to define Adams spectral sequence. This yields a topological characterization of the class of local spaces as the smallest $s$ containing $K(A, n)$'s and closed under homotopy inverse limit.

1. Introduction

Suppose that $R$ is a subring of the rational numbers or a finite field of the form $\mathbb{Z}/p\mathbb{Z}$, $p$ prime. In [1] Bousfield showed that any space $X$ has a functorial $R$-homology localization $R$; this is a space $X_R$ together with a map $X \to X_R$ which is terminal, up to homotopy, in the category of all maps $X \to Y$ that induce isomorphisms on mod $R$ homology. This paper proves a "Whitehead theorem" which is adapted to recognizing inverse limit constructions of $X_R$. In particular, the theorem shows that $X_R$ can be obtained from $X$ by transfinite iteration of an elementary homology approximation technique.

1.1. Organization of the paper. For background purposes, Section 2 describes in some detail our proposed construction for $X_R$. Section 3 contains some preliminary algebra, and Section 4 has a statement and proof of the Whitehead theorem itself. The last section shows how the Whitehead theorem can be used to demonstrate that the construction given in Section 2 actually works.

1.2. Remark. Section 2 is a straightforward attempt to transpose the algebraic towers of [2, §3, §8] into geometry. A less direct but more sophisticated...
inverse limit construction of $X_R$ appears in [4]. Our work here ultimately depends on a small but essential collection of Bousfield's algebraic lemmas from [2, §1–2, §6–7].

1.3. Notation and terminology. The word space is used as a synonym for simplicial set ([5], [6]). The symbol $R$ will always denote a fixed ring of the type described above; a space $X$ is said to be $R$-Bousfield if it is $H_*(-; R)$-local in the sense of [1, §1], that is, if the natural map $X \to X_R$ is a homotopy equivalence. Similarly, a group $\pi$ is called $R$-Bousfield if it is $HR$-local in the sense of [1, 5.1], and a $\pi$-module $M$ is $R$-Bousfield if it is $HR$-local as an (abelian) group and $HZ$-local [1, 5.3] as a $\pi$-module.

In these terms, theorem 5.5 of [1] reads that a connected space $X$ is $R$-Bousfield if $\pi_1X$ is $R$-Bousfield and the higher homotopy groups of $X$ are $R$-Bousfield as $\pi_1X$-modules.

2. A construction of $X_R$ by successive approximation

The idea of the construction will be to start with the simplest possible map (the map from $X$ to a one-point space) and iteratively modify the range of this map until an $R$-homology isomorphism is obtained.

2.1. The Dold-Kan construction. For any space $X$, $R \otimes X$ will denote the mod $R$ Dold-Kan construction on $X$, that is, $R \otimes X$ is the simplicial $R$-module which, for each $n \geq 0$, has as its set of $n$-simplices the free $R$-module on the $n$-simplices of $X$ [3, p. 14]. If $(Y, X)$ is a simplicial pair then $R \otimes (Y, X)$ will denote the quotient simplicial $R$-module pair $(R \otimes Y/R \otimes X, 0)$.

The homotopy groups of $R \otimes X$ are naturally isomorphic to the mod $R$ homology groups of $X$, and the natural inclusion $X \to R \otimes X$ induces a map on homotopy which is essentially the Hurewicz homomorphism. There are similar relative statements.

Suppose that $f : X \to Y$ is an arbitrary map of spaces. The mapping cylinder of $f$, denoted $\text{Cyl}(f)$, is defined as the pushout of the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{i_1} & & \downarrow{} \\
X \times \Delta[1] & \longrightarrow & \text{Cyl}(f)
\end{array}
$$

where $\Delta[1]$ is the standard 1-simplex [3, p. 234] and $i_1$ is the inclusion $x \mapsto (x, (1))$. The alternate inclusion $i_0 : X \to X \times \Delta[1]$ given by $x \mapsto (x, (0))$ induces an