LOCAL-GLOBAL PRINCIPLES
FOR ALGEBRAIC COVERS

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ABSTRACT
This paper is devoted to some local-global type questions about fields of
definition of algebraic covers. Let \( f: X \to B \) be a cover \emph{a priori} defined
over \( \mathbb{Q} \). Assume that the cover \( f \) can be defined over each completion \( \mathbb{Q}_p \)
of \( \mathbb{Q} \). Does it follow that the cover can be defined over \( \mathbb{Q} \)? This is the
local-to-global principle. It was shown to hold for \( G \)-covers [DbDo], i.e.,
for Galois covers given with their automorphisms. Here we prove that, in
the situation of mere covers, the local-to-global principle holds under some
additional assumptions on the group \( G \) of the cover and the monodromy
representation \( G \to S_d \) (with \( d = \deg(f) \)). This local-to-global problem
is closely related to the obstruction to the field of moduli being a field of
definition. This problem was studied in [DbDo], which is the main tool of
the present paper.

1. Presentation

1.1. THE LOCAL-TO-GLOBAL PROBLEM. Let \( B \) be an algebraic variety defined
over a number field \( K \) and \( f: X \to B \) be a cover \emph{a priori} defined over \( \overline{K} \).
Assume that the cover \( f \) can be defined over each completion \( K_v \) of \( K \). Does
it follow that the cover can be defined over \( K \)? We say that the \textit{local-to-global}
principle holds when the answer is "Yes". More generally, the same problem
can be considered with the base field \( K \) a field with a proper set \( M_K \) of places
satisfying the product formula. However, the local-to-global principle obviously
fails if the following condition does not hold:

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(*) The field $K$ is the only finite extension of $K$ which can be embedded in $K_v$ for all places $v \in M_K$.

Indeed, if $k$ is a proper extension of $K$ that can be embedded in all $K_v$'s, then a cover defined over $k$ but not on $K$ yields a counter-example to the local-to-global principle. We will assume that condition (*) holds. Classically, that is the case for number fields and for rational function fields $\overline{k}(T)$ in one indeterminate over an algebraically closed field.

Assumption (*) guarantees that if a cover is defined over each $K_v$, then its field of moduli is contained in $K$. Indeed, the field of moduli of a cover (§2.5) is the smallest possible field of definition, so is contained in each field of definition. However, the field of moduli need not be a field of definition. But when that is the case, the local-to-global principle obviously holds. For example, the field of moduli is a field of definition when the cover has no automorphisms (Fried [Fr]), or, when the cover is a Galois cover of $\mathbb{P}^1$ (Coombes–Harbater [CoHa]). The real problem is when the field of moduli is not a priori a field of definition.

The local-to-global problem was raised by E. Dew [Dew] for $G$-covers of $\mathbb{P}^1$ and for $K$ a number field. A $G$-cover is the data consisting of a Galois cover given together with its automorphisms. In the sequel, we use the phrase “mere covers” for non-necessarily Galois covers given without their automorphisms. Dew conjectured in particular that the local-to-global principle holds for $G$-covers of $\mathbb{P}^1$ over number fields. This was proved in [Db] except for number fields that are exceptions to Grunwald’s theorem (the field $\mathbb{Q}$ is not exceptional). This result was extended to $G$-covers of a general base space $B$ in [DbDo].

This paper is aimed at extending these results. The main direction is to consider the local-to-global principle for mere covers. We also systematically consider the situation where the base space $B$ is an arbitrary algebraic variety $B$ and $K$ is a more general field. These generalizations give rise to new difficulties that we explain below.

1.2. Main ingredients. The problem is closely related to the obstruction to the field of moduli being a field of definition. This is a classical problem, which was studied in quite a general way in [DbDo]: the Main Theorem of [DbDo] is a pure cohomological characterization of the obstruction. This will be the main tool of the paper.

1.2.1. Condition (Seq/Split): Essentially, the case that the base space $B$ is the