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ON HOPF ALGEBRAS AND RIGID MONOIDAL CATEGORIES

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ABSTRACT

Let $\mathscr{C}$ be a neutral Tannakian category over a field k. By a theorem of Saavedra Rivano there exists a commutative Hopf algebra A over k such that $\mathscr{C}$ is equivalent to the category of finite dimensional right A-comodules. We review Saavedra Rivano's construction of the bialgebra A and show that A has still an antipode if the symmetry condition on the monoidal structure of $\mathscr{C}$ is removed.

Introduction

Let k be a field, $\mathscr{C}$ a k-linear, abelian category which is essentially small, and let $\omega\colon \mathscr{C} \to \mathscr{V}ec_f(k)$ be a k-linear, exact and faithful functor from $\mathscr{C}$ into the category of finite dimensional k-vector spaces. By [3], p. 136, 2.6.3, there exists a k-coalgebra A such that ω factors through a k-linear equivalence $\mathscr{C} \to \mathscr{C}omod_f(A)$ of $\mathscr{C}$ with the category of finite dimensional right A-comodules. If in addition $\mathscr{C}$ has a rigid, symmetric monoidal structure (i.e. a symmetric monoidal structure such that every object has a dual object), and if ω is a symmetric monoidal functor, then A is a commutative Hopf algebra; it represents the functor

$$(1) \qquad T \mapsto \mathrm{Aut}^{\otimes}(\omega \otimes T)$$

which associates to a commutative k-algebra T the group of monoidal natural automorphisms of $\omega \otimes T\colon \mathscr{C} \to \mathscr{M}od(T)$, $X \mapsto \omega(X) \otimes T$, see [2], thm. 2.11, [3], II, 4.1.

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Suppose now that we drop from the above assumptions the symmetry condition on the monoidal structure of $\mathscr{C}$. Then A is still a bialgebra, which however does no longer represent the functor (1), because A may be non-commutative.

The aim of this note is to show that A is in fact still a Hopf algebra by deriving the existence of an antipode of A from the dual object functor of $\mathscr{C}$.

1. We first collect some facts on dual objects in monoidal categories, [1], [3]. Let $\mathscr{C}$ be a monoidal category, with product $\otimes$ and neutral object k. Let $X \in \mathscr{C}$. An object X^* of $\mathscr{C}$ is said to be a (left) dual of X if there exist morphisms

$$ev: X^* \otimes X \to k, \qquad \pi: k \to X \otimes X^*,$$

satisfying

$$id = (1 \otimes ev)(\pi \otimes 1): X \to X \otimes X^* \otimes X \to X,$$

$$id = (ev \otimes 1)(1 \otimes \pi): X^* \to X^* \otimes X \otimes X^* \to X^*.$$

Given such morphisms ev and π, it is easy to see that for all $Y, Z \in \mathscr{C}$ the maps

$$\mathrm{Hom}(Z \otimes X, Y) \to \mathrm{Hom}(Z, Y \otimes X^*), \qquad f \mapsto (f \otimes 1)(1 \otimes \pi),$$

$$\mathrm{Hom}(Z, Y \otimes X^*) \to \mathrm{Hom}(Z \otimes X, Y), \qquad g \mapsto (1 \otimes ev)(g \otimes 1)$$

are inverse to each other. Hence the functor $\mathscr{C} \to \mathscr{C}$, $Y \mapsto Y \otimes X^*$, is a right adjoint of $\mathscr{C} \to \mathscr{C}$, $Z \mapsto Z \otimes X$. Taking $Y = k$ shows X^* is uniquely determined up to isomorphism. More precisely, suppose that $(\overline{X}^*, \overline{ev}, \overline{\pi})$ is another dual of X. Define

$$\tau = (ev \otimes 1)(1 \otimes \overline{\pi}): X^* \to \overline{X}^*.$$

Then τ is an isomorphism and

$$ev = \overline{ev}(\tau \otimes 1), \qquad \overline{\pi} = (1 \otimes \tau)\pi. \tag{2}$$

Suppose now that $\mathscr{C}$ is rigid monoidal, i.e. every object $X \in \mathscr{C}$ has a dual X^*. Then for any morphism $f: X \to Y$ in $\mathscr{C}$ we can define

$$f^* = (ev \otimes 1)(1 \otimes f \otimes 1)(1 \otimes \pi): Y^* \to X^*.$$

This gives rise to a contravariant function $\mathscr{C} \to \mathscr{C}$, $X \mapsto X^*$, which is called the dual object functor of $\mathscr{C}$. Let $\omega: \mathscr{C} \to \mathscr{D}$ be a monoidal functor from $\mathscr{C}$ into a rigid monoidal category $\mathscr{D}$. Then $\omega(X^*)$ is naturally a dual object for $\omega(X)$. The corresponding τ is given by