A REMARK ON MUMFORD'S COMPACTNESS
THEOREM

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ABSTRACT
It is shown that a recent compactness theorem for Fuchsian groups, due to Mumford, remains valid for groups containing elliptic and parabolic elements.

A Fuchsian group $\Gamma$ is a discrete subgroup of the real Möbius group $G = SL(2, \mathbb{R})/\{ \pm I \}$. We will establish the following extension of a recent result by Mumford.

**THEOREM 1.** The set of conjugacy classes $[\Gamma]$ of Fuchsian groups $\Gamma$, such that $\operatorname{mes}(G/\Gamma) \leq \mu < \infty$ and the absolute value of the trace of every hyperbolic element $\gamma$ of $\Gamma$ is $\geq 2 + \varepsilon > 2$, is compact.

We recall that Fuchsian groups $\Gamma$ with $\operatorname{mes}(G/\Gamma) < \infty$ are finitely generated. Hence the space of conjugacy classes $[\Gamma]$ of all such groups has a natural topology: a (distinguished) neighborhood $V$ of $\Gamma$ is determined by a sequence $\{ \gamma_1, \ldots, \gamma_r \}$ of generators of $\Gamma$ and a neighborhood $v$ of the identity in $G$; a conjugacy class $[\Gamma']$ belong to $V$ if and only if there is an isomorphism $\chi$ of $\Gamma$ onto a $\Gamma'' \in [\Gamma']$ such that $\chi(\gamma)$ is parabolic if and only if $\gamma$ is, and $\chi(\gamma_j) \circ \gamma_j^{-1} \in v$ for $j = 1, \ldots, r$.

In [5] Mumford proved a general compactness theorem and obtained, as a corollary, a statement analogous to Theorem 1, under the additional hypotheses that all groups $\Gamma$ considered are torsion free and all quotients $G/\Gamma$ are compact. He stated that the corollary can be obtained by an elementary argument. Our proof of Theorem 1 is an extension of this argument.

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A Fuchsian group $\Gamma$ acts on the upper half plane $U$ as a group of conformal automorphisms; the condition $\text{mes} \, (G/\Gamma) \leq \mu$ is equivalent to the condition $\iint_{U/\Gamma} y^{-2} \, dx \, dy \leq c \mu$ where $c$ is a universal constant. Every group $\Gamma$ satisfying this condition has a signature

$$\sigma = (p, n; v_1, \ldots, v_n)$$

where $p$ and $n$ are integers, the $v_j$ are integers or the symbol $\infty$, and

$$p \geq 0, \quad n \geq 0, \quad 2 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \infty,$$

$$A(\sigma) = 2\pi(2p - 2 + n - \frac{1}{v_1} - \cdots - \frac{1}{v_n}) > 0.$$ 

We have that $\iint_{U/\Gamma} y^{-2} \, dx \, dy = A(\sigma)$, the Riemann surface $U/\Gamma$ is a compact surface of genus $p$ with $n_\infty$ points removed, $n_\infty$ being the number of times $\infty$ occurs among the symbols $v_1, \ldots, v_n$, and $\Gamma$ has precisely $n$ non-conjugate in $\Gamma$ maximal cyclic elliptic or parabolic subgroups, the order of these subgroups being $v_1, \ldots, v_n$. Note that $U/\Gamma$ is compact if and only if $G/\Gamma$ is, and if and only if $n = 0$ or $v_n < \infty$; we call such signatures of compact type.

A Fuchsian group with signature (1) is said to represent the configuration

$$\Sigma = (S; P_1, \ldots, P_n)$$

where $S$ is a compact Riemann surface of genus $p$ and $P_1, \ldots, P_n$ are distinct points on $S$, if there is a conformal bijection $f: U/\Gamma \to S - \{P_1, \ldots, P_n\}$ such that $f^{-1}(P_j)$ is the image under $U \to U/\Gamma$ of a point $z_j \in U$ fixed under a maximal cyclic subgroup of $\Gamma$ of order $v_j$, $j = 1, \ldots, n - n_\infty$. The group determines the configuration $\Sigma$ except for a conformal equivalence and a permutation of the "ramification points" $P_j$ in which each $P_i$ is taken into $P_k$ with $v_i = v_k$. Conversely, given $\sigma$ and $\Sigma$, satisfying (1), (2) and (3), there is a Fuchsian group $\Gamma$ of signature $\sigma$, determined up to conjugacy in $G$, which represents $\Sigma$. This is the limit circle theorem of Klein and Poincaré.

We denote by $X(\sigma)$ the set of conjugacy classes $[\Gamma]$ of Fuchsian groups $\Gamma$ with signature $\sigma$. (If $\sigma = (p, 0)$ $p > 1$, then $X(\sigma)$ is the space of moduli of compact Riemann surfaces at genus $p$.) One verifies, for instance by using quasiconformal mappings, that the spaces $X(\sigma)$, with their natural topologies, are metrizable. More precisely, the topology of $X(\sigma)$ can be derived from the Teichmüller metric defined as follows: the distance between $[\Gamma]$ and $[\Gamma']$ is the smallest number $\alpha$