THE CONSTRUCTION OF QUASI-ININVARIANT
MEASURES

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ABSTRACT
If a homeomorphism possesses a non-atomic ergodic measure it has recurrent
points. If it has recurrent points then there are uncountably many inequivalent
ergodic quasi-invariant measures.

There has recently been some interest expressed in the construction of many
inequivalent quasi-invariant ergodic measures for uniquely ergodic systems.
M. Keane, in [1], has given an explicit construction of uncountably many inequiva-
 lent such measures for the irrational rotations of the circle while W. Krieger, [2],
has applied category methods to obtain such collections for any uniquely ergodic
homeomorphism. We propose to describe a simple construction of such families
of measures under quite general conditions, that are in fact necessary. For com-
pleteness sake we review the definitions. We will be dealing with a fixed compact
metric space X and a homeomorphism \( \phi \) of X onto itself. All measures will
be finite measures defined on the Borel field \( \mathcal{B} \) of X. A measure \( \mu \) is said to be ergodic if

\[
E \in \mathcal{B}, \ \phi E = E \ \text{implies} \ \mu(E) \mu(X \setminus E) = 0.
\]

It is said to be a quasi-invariant measure (q.i.) if

\[
\mu(E) = 0 \ \text{iff} \ \mu(\phi E) = 0.
\]

Since any atomic measure whose support is the orbit of a single point is an ergodic
q.i. we promptly lose interest in such measures, and concentrate on finding non-
atomic ergodic q.i.'s. The following is a necessary condition for the existence of
non-atomic ergodic measures.

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LEMMA 1. If μ is a non-atomic ergodic measure then μ—a.e. point of X is recurrent (i.e. φ^n x returns infinitely often to any deleted neighborhood of x).

PROOF. Let \( A_\varepsilon = \{ x : \inf_{n \geq 0} d(\phi^n x, x) \geq \varepsilon \} \), where d is a metric on X. Since \( \mu \) —a.e. point is non periodic (ergodicity), it suffices to show that \( \mu(A_\varepsilon) = 0 \). Suppose that \( \mu(A_\varepsilon) > 0 \), then there is an \( A \subset A_\varepsilon \) with \( \text{diam}(A) \leq \varepsilon /3 \) and \( \mu(A) > 0 \). Observe that, for \( n \neq 0 \), \( \phi^n A \cap A = \emptyset \). Since \( \mu \) is non-atomic there is a \( B \subset A \) with \( \mu(B) \mu(A \setminus B) > 0 \). But then \( \bigcup_n \phi^n B = E \) is invariant and disjoint from \( A \setminus B \), thus \( \mu(E) \mu(X \setminus E) > 0 \) contradicting the ergodicity. □

The existence of some recurrent point is all that we shall need for our construction of uncountably many inequivalent ergodic q.i.'s. The next simple observation means that we can forget about the quasi-invariant part.

Given a measure \( \mu \) define \( \phi^n \mu \) by \( \phi^n \mu(E) = \mu(\phi^n E) \). Let \( c_n > 0 \), with \( \sum_{-\infty}^{\infty} c_n = 1 \), then

\[
\bar{\mu} = \sum_{-\infty}^{\infty} c_n \phi^n \mu
\]

is a q.i. and indeed the "minimal" q.i. with respect to which \( \mu \) is absolutely continuous.

For later reference note that if \( \mu \) is ergodic so is \( \bar{\mu} \).

LEMMA 2. Let \( \{ \mu_\varepsilon \} \) be an uncountable collection of mutually singular non-atomic measures. Then \( \{ \mu_\varepsilon \} \) contains an uncountable family of mutually singular measures.

PROOF. The lemma clearly follows if we show that for \( \mu \in \{ \mu_\varepsilon \} \) there are at most countably many \( \nu \in \{ \mu_\varepsilon \} \) such that \( \bar{\nu} \), \( \bar{\mu} \) are not mutually singular. This in turn follows if we show that for each integer \( n \), there are at most countably many \( \nu \)'s which are not singular with respect to \( \phi^n \mu \). If \( \nu \) is not singular with respect to \( \phi^n \mu \) write \( \nu = v_c + v_s \) where \( v_c \neq 0 \) is the absolutely continuous (\( \phi^n \mu \)), i.e., \( v_c \in L^1(\phi^n \mu) \). Since the \( v \)'s are mutually singular so are the \( v_c \)'s which means that they have disjoint supports (which are sets of positive \( \phi^n \mu \) measure) and their number is therefore finite or countable. □

One final preliminary lemma is well known, a proof is included for the reader's convenience.

LEMMA 3. Suppose that for arbitrarily large \( N \) there are disjoint sets \( A_1^{(N)}, A_2^{(N)}, \ldots, A_N^{(N)} \) such that \( \mu(X \cup A_i^{(N)}) = 0 \) and for any \( i,j \) there is an \( n = n(i,j) \) and \( c = c(i,j) > 0 \) such that \( \phi^n A_i^{(N)} = A_j^{(N)} \) and for \( E \subset A_i^{(N)} \), \( \mu(E) \)