CONSTRUCTING LIE ALGEBRAS OF FIRST-ORDER DIFFERENTIAL OPERATORS

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Formulas for calculating vector fields — generators of groups of transformations to a uniform space — from specified structural constants are obtained. The problem of vector-field continuation — the construction of Lie algebras of inhomogeneous first-order differential operators — is considered. It is also shown that the existence of a nontrivial continuation is closely associated with the structure of the isotopic subalgebra and, in particular, that no nontrivial continuation exists for semisimple algebras.

INTRODUCTION

In many problems of mathematical physics, the action of a local Lie group on a homogeneous space of specified dimensionality must be derived on the basis of the structural constants of the Lie algebra or, alternatively, the construction of a Lie algebra of vector fields in this homogeneous space. In particular, the basic element of the method of noncommutative integration of linear partial differential equations [1] is the construction of a so-called representation of a Lie algebra — the realization of Lie algebras by first-order differential operators of special form. (A precise definition is given in what follows.)

This problem is also relevant to the classification of the metrics of Riemann spaces with respect to groups of motions. Metric classification may be approached as follows: for the given Lie algebra, all its nonequivalent (i.e., differing in any coordinate transformations) realizations of the Killing vector are established, and then the metrics are found from the corresponding Killing equations. The classification of four-dimensional Riemann spaces with respect to groups of motions was solved by a slightly different method, with broader use of geometric ideas, in [2-4]. (The basic content of this work is summarized in [5].) The basis of this approach is a theorem proven by Lie [6].

THEOREM 1. Two r-parametric groups (transformations) of identical structure, containing the same number of variables, for which the common ranks of the matrices \( \xi \) and \( \xi' \) are less than \( r \), are similar if and only if the ranks are equal (say, \( q \)), any pair of corresponding minors of order \( q \) have the same ranks, and the corresponding system of equations

\[
\phi^A_{\rho}(x') = q^A_{\rho}(x), \ h = 1, ..., q, \ \rho = q + 1, ..., r,
\]

is consistent and does not lead to a relation between the variables of any group.

Here \( \xi_i = \xi_i(x) \partial_i x, \ \xi'_j = \xi'_j(x') \partial_i x' \) are generators of the corresponding groups of transformations: \( \xi^\alpha = \phi^A_{\rho} \xi^\rho_{\alpha}/(\rho = q + 1, ..., n) \).

Note that this theorem does not give a method of constructing Lie algebras of vector fields that are known to be nonequivalent.

The approach used in the present work is based on the well-known mutually unique correspondence between points of a homogeneous space \( M \) of group \( F \) and the right (or left) adjacent classes \( H \backslash G \), where \( H \) is the isotopic subgroup (the action of the group \( G \) to the right) [7]. One of the basic results of the present work is the derivation of explicit formulas for calculating vector fields — generators of a group of transformations for uniform space — from specified structural constants. It is shown here that two Lie algebras of vector fields with the same structural constants are equivalent if and only if the corresponding isotopic subgroups are conjugate. As an illustration, all the nonequivalent realizations of the Lie algebras of the
generators of the corresponding group of transformations to a three-dimensional uniform space are found for one four-dimensional Lie algebra. The metric of the four-dimensional Riemann space with this group of transformations is found. (It was asserted in [5], pp. 283, 287, that there are no Riemann spaces with this group of motions.)

The problem of vector-field continuation — the construction of Lie algebras of inhomogeneous first-order differential operators — is considered in a separate section. A particular case of this problem is the construction of \( \lambda \) representations, which is possible for an arbitrary \( \lambda \) algebra, as shown in the present work. It is also shown that the existence of a nontrivial continuation is closely related to the structure of the isotopic subalgebra and, in particular, no nontrivial continuation exists for semisimple algebras.

1. ALGORITHM FOR CONSTRUCTING LEFT-INARIANT VECTOR FIELDS AND DIFFERENTIAL FORMS

Suppose that \( V \) is an open vicinity of a single \( n \)-dimensional Lie group \( G \). and \( \psi \) is a homomorphism of \( V \) on the open set \( U \) of Euclidean space \( \mathbb{R}^n \), i.e., \((V, \psi)\) is a map on \( G \). In the present work, only local properties will be studied, and therefore it is everywhere assumed that all the vector fields for group \( G \) and all of its elements considered and their products lie in region \( V \). All the functions encountered are also assumed to be analytic.

Let's introduce some notation. Each element of the group from region \( V \) is uniquely determined by its coordinates. This dependence will be indicated in explicit form: \( g_z = \psi^{-1}(z) \in V \); here \( z = (z^1, \ldots, z^n) \in U \). For the product of group elements, we have: \( g_ag_z = g_{u} \); \( u = \varphi(a, z) \). Here \( \varphi \) is the composition function; \( g_{a}, g_{z}, g_{u} \in V \). It is also assumed that the unit element corresponds to zero coordinate values: \( \psi(e) = 0 \). The tangential vectors \( \partial_{a} \) at point \( z \) form the basis of the tangential space \( T_z \mathbb{R}^n \) at this point. The corresponding tangential vectors \((\psi_{*}^{-1})\partial_{a} = \partial_{a}g_{z} \in T_{g_z}G \) form the basis of the tangential space to group \( G \) at point \( g_z \). Setting \( z = 0 \), the tangential vectors \( \partial_{a}g_{z} \mid_{z=0} = e_{a} \) form the basis of the Lie algebra \( L \) of the local group \( G \) with commutation relations \([e_{a}, e_{b}] = C_{ab}^{c}e_{c} \).

Let \( \xi_{a} = \xi_{a}^{i}(z)\partial_{a} \) denote the left-invariant vector fields for group \( G \) in coordinate representation. The left-invariant vector field \( \xi_{a} \) is the generator of a right regular representation and is determined by the formula

\[
\xi_{a}^{i}(z) = \frac{\partial \xi^{i}(z, a)}{\partial a^{i}} \bigg|_{a=0}.
\]

The left invariance of the fields \( \xi_{a} \) implies an identity on the basis of which the third Lie theorem (the construction of a local Lie group from structural constants) is proven

\[
\xi_{a}^{i}(z) \frac{\partial \xi^{j}(a, z)}{\partial z^{j}} = \xi_{a}^{j}(z(a, z)).
\]

We introduce the left-invariant unit form \( \omega^{a} = \omega^{a}(z)dz^{a} \) for the Lie group. It satisfies the Maurer – Kähler equation \( d\omega^{a} = -(1/2)C_{ab}^{c}\omega^{b} \wedge \omega^{c} \). The matrix of coefficients \( \omega_{a}^{i} \) of the unit form is the inverse of the matrix \( \xi_{a}^{i}: \omega_{a}^{i}(z)\xi_{a}^{j}(z) = \delta_{a}^{j} \).

Our immediate goal is to derive the vector fields \( \xi_{a} \) from the structural constants \( C_{ab}^{c} \). Applying the left-shift differential \((L_{g_{z}})_{*}\) to the basis vector of the Lie algebra \( e_{a} \), we obtain

\[
(L_{g_{z}})_{*} e_{a} = (L_{g_{z}})_{*} \left[ \partial_{a} \xi_{a}^{i}(z) \mid_{a=0} \right] \xi_{a}^{i}(z)g_{z} = \partial_{a} \xi_{a}^{i}(z)g_{z} = \xi_{a}^{i}(z) \partial_{a}g_{z} = \omega_{a}^{i}(z) \partial_{a}g_{z} = \omega_{a}^{i}(z) \partial_{a}g_{z} \mid_{a=0} = \omega_{a}^{i}(z) \partial_{a}g_{z} \mid_{a=0} = \omega_{a}^{i}(z) \partial_{a}g_{z}.
\]

which is equivalent to

\[
\omega_{a}^{i}(z) c_{a} = (L_{g_{z}}^{-1})_{*} \partial_{a}g_{z}.
\]