SIMPLIFIED EQUATIONS FOR INVESTIGATION
OF CHARGED-PARTICLE DYNAMICS IN
MAGNETIC FIELDS

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A system of motion equations presented for charged particles in betatron magnetic fields makes it possible to conduct all particle dynamics calculations in relative units because the particle velocity components are expressed in terms of trajectory geometry using comparatively simple notation, and the conversion from relative to absolute units is easily carried out in terms of the radius of a fixed orbit and the magnetic induction of the field on it. This equation system has been used successfully to investigate particle dynamics in an electron beam extracted from a betatron and to determine the principal parameters of the beam; the results were confirmed in practice.

It is necessary in the development of cyclotron charged-particle accelerators and beam transport and shaping systems to investigate the behavior of the particle beam as it passes through a certain magnetic system. Usually, beam-particle dynamics is investigated with the aid of simplified equations of motion. The greatest difficulty encountered here is with the method used to eliminate the time coordinate from the motion equations in order to convert to dimensionless parameters. Analysis of published papers [1, 2] on particle dynamics indicates that the usual practice has been to simplify by assuming that the particle-revolution frequency is constant in the accelerator field, the total velocity of each beam particle being equated to the velocity component of the motion on some fixed orbit, which is often referred to as the equilibrium orbit. These assumptions are acceptable only if the deviations from the orbit are small, and are not suitable for description of large oscillation amplitudes.

On the other hand, investigation of charged-particle dynamics with the complete equations of motion is very complicated owing to the need to operate at all times with parameters expressed in absolute units.

Below we derive equations of motion that are not subject to the above deficiencies.

The equations of motion of an electron in a magnetic field in the projections onto the cylindrical axes r, \(\varphi\), z are quite well known and can be derived from the Lagrange equations [3]. Recognizing that the magnetic field does not change the energy of the electron, time can be eliminated from the equations of motion by using the relation that links the particle’s oscillation energies on the coordinate axes:

\[
V_0^2 = r^2 + (r\varphi)^2 + z^2,
\]

where \(V_0\) is the velocity of the particle and the dots over \(r\), \(\varphi\), and \(z\) indicate differentiation with respect to time \(t\). Appropriate manipulation yields a system of two second-order differential equations [4]:

\[
r'' - 2 \frac{r'^2 - r}{r^2} - r = - \frac{e \cdot c^2}{E \cdot V_0} \cdot (r'^2 + r^2 + z'^2)^{1/2} \times \left[ \frac{B_z}{r} \cdot (r'^2 + r^2) - z' \cdot B_\varphi - z' \cdot \frac{r'}{r} \cdot B_r \right],
\]

\[
z'' - 2 \frac{r'^2 z' - z}{r^2} = - \frac{e \cdot c^2}{E \cdot V_0} \cdot (r'^2 + r^2 + z'^2)^{1/2} \times
\]


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where the prime indicates differentiation with respect to \( \phi \), \( e \) and \( E \) are the charge and relativistic energy of the electron, respectively, \( c \) is the velocity of light in vacuum, and \( B_r, B_\phi, \) and \( B_z \) are the components of the magnetic induction vector on the corresponding axes. We note that the total particle velocity is related to the radius \( r_0 \) of the fixed orbit and the induction \( B_0 \) on that orbit by the expression

\[
V_0 = \frac{e^2}{c^2}B_0 \cdot r_0 = \omega_0 \cdot r_0,
\]

where \( \omega_0 \) is the particle’s revolution frequency on the fixed orbit. All of the various modifications of the equations of particle motion in a magnetic field that have been used previously, including the well-known betatron oscillation equations [2, 5, 6], can be derived from equation system (2) by imposing various limits on the oscillation amplitude or on the particle revolution frequency. It will be understood that such limitations distort the true picture of the particle in the magnetic field and lead to errors.

Equation system (2) can be reduced to a simple form that makes it possible to conduct all calculations in relative units if the particle-velocity components are expressed in terms of trajectory geometric parameters using the notation

\[
\begin{align*}
V_\phi &= \frac{rd\phi}{dt} = V_0 \cdot \sin \alpha \cdot \cos \gamma, \\
V_r &= \frac{dr}{dt} = V_0 \cdot \cos \alpha \cdot \cos \gamma, \\
V_z &= \frac{dz}{dt} = V_0 \cdot \sin \gamma.
\end{align*}
\]

Here \( V_\phi, V_r, \) and \( V_z \) are the components into which the total velocity \( V_0 \) is decomposed on the \( \phi, r, \) and \( z \) axes, as indicated in Fig. 1, \( \alpha \) is the angle between the direction of the electron-velocity projection onto the plane of the fixed orbit and the direction of the radius vector \( r, \) and \( \gamma \) is the angle between the direction of the velocity \( V_0 \) and that of its projection onto the plane of the fixed orbit.

Using the notation of (4), we obtain a system of four first-order differential equations from system (2):

\[
\begin{align*}
\frac{d\alpha}{d\phi} &= \frac{1}{r_0 \cdot B_\phi} \left[ \frac{r \cdot B_z}{\sin \alpha \cdot \cos \gamma} - \frac{r \cdot B_\phi \cdot \sin \gamma}{\cos^2 \gamma} - \frac{r \cdot B_z \cdot \cos \alpha \cdot \sin \gamma}{\sin \alpha \cdot \cos^2 \gamma} \right], \\
\cos \gamma \frac{dz}{d\phi} &= \frac{1}{r_0 \cdot B_\phi} \left[ r \cdot B_z - r \cdot B_\phi \cdot \frac{\cos \alpha}{\sin \alpha} \right], \\
\frac{dr}{dz} &= \frac{r \cdot \cos \alpha}{\sin \alpha}, \\
\frac{dz}{dz} &= \frac{r \cdot \sin \gamma}{\sin \alpha \cdot \cos \gamma}.
\end{align*}
\]