NEIGHBORHOODS OF EXTREME POINTS(1)

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ABSTRACT

An examination of relationship between two neighborhood systems (relative to two linear topologies) of extreme points yields a unified approach to some known and new results, among which are Bessaga-Pelczynski's theorem on closed bounded convex subsets of separable conjugate Banach spaces and Ryll-Nardzewski's fixed point theorem.

§0. Introduction. Let $C$ be a compact subset of a Banach space $E$. Then, of course, the norm topology and the weak topology agree on $C$. Now suppose that $C$ is only weakly compact. Then the identity map: $(C, \text{weak}) \rightarrow (C, \text{norm})$ is no longer continuous in general. Nevertheless one may still ask how the set of points of continuity of this map is distributed in $C$. In particular, when $C$ is convex as well as weakly compact, is the identity map: $(C, \text{weak}) \rightarrow (C, \text{norm})$ continuous at any of the extreme points of $C$, i.e., do there exist extreme points of $C$ which have weak neighborhoods (relative to $C$) of arbitrarily small diameter? The importance of an answer to such a question is demonstrated in Rieffel [7] and in note [6]. Professor J. L. Kelley also recognized the relevance of this question to Ryll-Nardzewski's fixed point theorem. The work of Lindenstrauss in [4] yields the following answer: if $C$ is a weakly compact, convex subset of a separable Banach space, then there are "many" extreme points of $C$, where the identity map $(C, \text{weak}) \rightarrow (C, \text{norm})$ is continuous. This fact was proved by using deep Banach space techniques due to Kadec and Lindenstrauss. In the present article, we shall generalize this result in various directions. The main theorem of this paper (Theorem 2.3) is stated in somewhat obscure, if not pedantic, language, because we tried to combine all the generalizations into one theorem. However, we hope this is forgiven because of the diverse applications of the single theorem. Here are some of the consequences of the main theorem: each bounded subset of a separable, conjugate Banach space is "dentable" in the sense of [7]; each closed, convex, bounded subset of $E$ is the closed convex hull of its extreme points, where $E$ is either a separable, conjugate Banach space or a Fréchet space such that $E^{**}$ is separable relative to its strong topology. In addition, a slight generalization of Ryll-Nardzewski's fixed point theorem can easily be derived from the main theorem.

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The paper is organized as follows: Section §0, the present one, is the introduction. Section §1 contains the preliminary material, and section §2 is devoted to the main theorem. Our proof of the main theorem is independent of Lindenstrauss' work and is quite different in spirit. Category plays a large rôle throughout §§1-2. Section §3 gives applications of the main theorem. Our terminology and notation will be those of Kelley, Namioka, et al. [3].

Finally, we wish to thank R. Phelps for many enlightening discussions on the subject of the present paper.

1. Preliminaries. Let \((E,\mathcal{T})\) be a linear topological space, and let \(A\) be a subset of \(E\). Then we denote by \((A,\mathcal{T})\) the space \(A\) with the topology induced by \(\mathcal{T}\). If \(p\) is a pseudo-norm on the linear space \(E\), then \(\mathcal{T}_p\) denotes the pseudo-norm topology on \(E\) given by \(p\). The pseudo-norm \(p\) is lower \(\mathcal{T}\)-semicontinuous if \(\{x: p(x) \leq 1\}\) is \(\mathcal{T}\)-closed. If \(V\) is a convex, circled, \(\mathcal{T}\)-closed subset of \(E\), which is radial at 0,\(^2\) then the Minkowski functional of \(V\) is lower \(\mathcal{T}\)-semicontinuous. For instance, if \(E\) is a normed linear space, then the norm on \(E\) is lower \(w\)-semicontinuous, and the norm on the dual \(E^*\) is lower \(w^*\)-semicontinuous, where \(w\) and \(w^*\) are the topologies \(w(E, E^*)\) and \(w(E^*, E)\) respectively.

1.1 Lemma. Let \(X\) be a compact Hausdorff space, and let \(\{C_i: i = 1, 2, \ldots\}\) be a sequence of closed subsets of \(X\) such that \(X = \bigcup\{C_i: i = 1, 2, \ldots\}\). Then \(\bigcup\{\text{Int}C_i: i = 1, 2, \ldots\}\) is dense in \(X\), where \(\text{Int}C_i\) is the interior of \(C_i\) in \(X\).

Proof. We may assume that \(X \neq \emptyset\). Let \(U\) be an open nonempty subset of \(X\). Then \(U\) is locally compact, and hence \(U\) is of the 2nd category in itself. Since \(U = \bigcup\{U \cap C_i: i = 1, 2, \ldots\}\) and \(U \cap C_i\) is closed in \(U\), for at least one \(i\), \(U \cap C_i\) has non-empty interior relative to \(U\) and hence relative to \(X\). Therefore, \(U \cap \bigcup\{\text{Int}C_i: i = 1, 2, \ldots\}\) is dense in \(X\), and since, \(U\) is arbitrary, \(\bigcup\{\text{Int}C_i: i = 1, 2, \ldots\}\) is dense in \(X\).

1.2 Proposition. Let \((E, \mathcal{T})\) be a Hausdorff linear topological space, let \(p\) be a lower \(\mathcal{T}\)-semicontinuous pseudo-norm such that \((E, \mathcal{T}_p)\) is separable, and let \(K\) be a \(\mathcal{T}\)-compact subset of \(E\). The set of all points of continuity of the identity map: \((K, \mathcal{T}) \to (K, \mathcal{T}_p)\) is a dense \(G_\delta\) subset of \((K, \mathcal{T})\).

Proof. For a subset \(X\) of \(E\), define \(p\)-diam(\(X\)) = \(\sup\{p(x - y): x, y \in X\}\). For each \(\varepsilon > 0\), let \(A_\varepsilon\) be the union of all open subsets of \((K, \mathcal{T})\) of \(p\)-diam \(\leq \varepsilon\). Clearly \(A_\varepsilon\) is open. Let \(S = \{x: p(x) \leq \varepsilon/2\}\). Then, since \((E, \mathcal{T}_p)\) is separable, there is a sequence \(\{x_i\}\) in \(E\) such that \(K = \bigcup\{K \cap (x_i + S): i = 1, 2, \ldots\}\), and each \(K \cap (x_i + S)\) is \(\mathcal{T}\)-closed because \(p\) is lower \(\mathcal{T}\)-semicontinuous. Hence, by Lemma 1.1, the union of the interiors of \(K \cap (x_i + S)\) in \((K, \mathcal{T})\) is dense in \(K\), and this union is clearly contained in \(A_\varepsilon\). Therefore \(A_\varepsilon\) is a dense open subset

\(^2\) \(V\) is radial at 0, if, for each \(x\) in \(E\), there is a positive number \(t\) such that \(sx \in V\) whenever \(0 \leq s \leq t\).