THE INCREASE OF SUMS AND PRODUCTS DEPENDENT ON \((y_1, \ldots, y_n)\) BY REARRANGEMENT OF THIS SET

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ABSTRACT

Let \(F(u, v)\) be a symmetric real function defined for \(\alpha < u, v < \beta\) and assume that \(G(u, v, w) = F(u, v) + F(u, w) - F(v, w)\) is decreasing in \(v\) and \(w\) for \(u \leq \min(u, v)\). For any set \(\{y_1, \ldots, y_n\}\), \(\alpha < y_i < \beta\), given except in arrangement \(\sum_{i=1}^{n} F(y_i, y_{i+1})\) where \(y_{n+1} = y_1\) is maximal if (and under some additional assumptions only if) \(\{y_i\}\) is arranged in circular symmetrical order. Examples are given and an additional result is proved on the product \(\prod_{i=1}^{n} (y_1 y_2 + a_1 (y_1 y_2)^2 + \ldots + a_n)\) where \(a_n \geq 0\) and where the set \(\{y_1, \ldots, y_n\}\), \(y_i \geq 0\) is given except in arrangement. The problems considered here arose in connection with a theorem by A. Lehman [1] and a lemma of Duffin and Schaeffer [2].

We start with some definitions given in [1].

The sets \((y^-) = (y_1^-, y_2^-, \ldots, y_n^-)\) and \((-y) = (-y_1, \ldots, -y_n)\) are symmetrically decreasing rearrangements of an ordered set \(\{y_1, \ldots, y_n\}\) of \(n\) real numbers if

\[
y_1^- \leq y_2^- \leq \cdots \leq y_n^-
\]

and

\[
y_n^- \leq -y_1^- \leq -y_{n-1}^- \leq \cdots \leq -y_{(n+1)/2}^-.
\]

A circular rearrangement of an ordered set \(\{y_1, \ldots, y_n\}\) is a cyclic rearrangement of \(\{y_i\}\) or a cyclic rearrangement followed by inversion.

An ordered set \(\{y_1, \ldots, y_n\}\) of \(n\) real numbers is arranged in circular symmetrical order if one of its circular rearrangements is symmetrically decreasing. It follows that the sets \((y^-), (-y)\) are arranged in circular symmetrical order and so is the set \(\{y_1, \ldots, y_n\}\) if either

\[
y_1 \leq y_2 \leq \cdots \leq y_{(n+3)/2}
\]

or

\[
y_2 \leq y_1 \leq \cdots \leq y_{(n+4)/2}
\]

Received Aug. 3, 1966, and in revised form April 27, 1967.

This paper is part of the author's Master of Science dissertation at the Technion-Israel Institute of Technology.

The author wishes to thank Professor B. Schwarz and Professor E. Jabotinsky for their help in the preparation of this paper.
Theorem 1. Let \( F(u,v) \) be a symmetric real function defined for 
\[ \alpha < u, v < \beta \quad -\infty \leq \alpha < \beta \leq \infty, \]
and assume that the function 
\[ G(u,v,w) = F(u,v) + F(u,w) - F(v,w), \quad \alpha < u,v,w \leq \beta \]
is decreasing in \( v \) and \( w \) for \( u \leq \min(v,w) \).

Let the set \( (y) = (y_1, \ldots, y_n) \alpha < y_i < \beta, i = 1, \ldots, n \) be given except in arrangement. Then

\[ S_n = \sum_{i=1}^{n} F(y_i, y_{i+1}), \quad (y_{n+1} = y_1) \]
is maximal if \( (y) \) is arranged in circular symmetrical order.

Moreover, if \( G(u,v,w) \) is strictly decreasing in \( v \) and \( w \) for \( u < \min(v,w) \) and 
\[ F(u,u) = G(u,v,u) = G(u,u,w) > G(u,v,w) \quad u < \min(v,w) \]
and if, in addition, no three elements of \( (y) \) have the same value, then (6) attains its maximum only if \( (y) \) is arranged in circular symmetrical order.

Proof. As the proof is similar to the proof in [1], we give here only a short outline.

The first assertion of the theorem is equivalent to

\[ S_n^- = \sum_{i=1}^{n} F(y_i, y_{i+1}) \geq \sum_{i=1}^{n} F(y_i, y_{i+1}) = S_n \]
(8)
\[ y_{n+1} = y_1, y_{n+1}^- = y_{i+1}, y_i \leq y_{i+1}, i = 2, \ldots, n. \]

(8) is proved by induction, using the equalities

\[ \sum_{i=1}^{n} F(y_i, y_{i+1}) = \sum_{i=1}^{n-1} F(x_i, x_{i+1}) + G(y_n, x_{i+1}, x_{i-1}) \]
(9)
\[ \sum_{i=1}^{n} F(x_i, x_{i+1}) + G(y_n, x_1, x_{n-1}) \]
\[ y_{n+1} = y_1, x_n = x_1, y_{n+1}^- = y_1, x_n^' = x_1^' \]
where \( x_i = y_{i+1}, x_i^' = y_{i+1}, i = 1, \ldots, n - 1 \) (hence \( x' = (-x) \)).

The second assertion is also proved by induction and (9) again allows us to proceed from \( n - 1 \) to \( n \).

For the sets \((1,2,3,4,2,2)\) and \((1,2,2,3,4,2)\) and any symmetric function \( F \) the sums (6) are equal; hence it is necessary for the second part of the theorem to assume that no three elements of \((y)\) have the same value.