RIEMANN FUNCTIONS
FOR A SYSTEM OF HYPERBOLIC FORM
IN THREE INDEPENDENT VARIABLES

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ABSTRACT
Functions are defined which permit the solution of a special hyperbolic
system to be expressed as a quadrature of its initial data over the initial
surface.

1. Introduction. In this paper Riemann functions (R.F.) are defined for
systems of partial differential equations of the type

\[ L(U) = (D - A) U = 0 \]

where

\[ U \equiv (U^1, \ldots, U^N), \quad A \equiv (a_{ij}(x)), \quad x \equiv (x_1, x_2, x_3) \]

\[ D \equiv (D_1, \ldots, D_N), \quad D_i = \sum_{j=1}^{3} \alpha_{ij} \frac{\partial}{\partial x_j} \]

and the direction numbers \( \vec{\alpha}_i \equiv (\alpha_{i1}, \alpha_{i2}, \alpha_{i3}) \) are constant, distinct and oblique to the
initial data surface.

We assume that initial data is specified on an initial data surface, \( \theta \), and for
simplicity and without loss of generality chose for \( \theta \) the hyperplane \( x_1 = 0 \). We
shall show that the value of \( U \) at any point \( P \), not on \( x_1 = 0 \) is a quadrature
of its initial data and the R.F. over a subset of the initial hyperplane.

Also for purposes of simplicity and visualization the further non-restrictive
hypothesis are made that \( P \) is in the upper half plane, and that the vectors \( \vec{\alpha}_i \),
\( i = 1, \ldots, N \) are direction cosines and have positive projections in the \( x_1 \) direction
i.e. the vectors \( \vec{\alpha}_i \) point in the general direction of the positive \( x_1 \) axis. A restrictive
assumption we make is that no three vector \( \vec{\alpha}_i \) through \( P \) are coplanar; this as-
sumption will finally be removed.

Riemann functions were defined for systems similar to (1.1) in [1, 2, 3] and the
techniques used here are a synthesis of ideas introduced in those papers. Although this paper is almost completely self-contained, a familiarity with [1, 2, 3] should be helpful.

2. Orientation and notation. The R.F. for each component of $U$ is a set of vector valued functions. Each member of the set is a solution to the adjoint operator to (1.1), $L^*$, defined in a domain $D^i$. These domains are three dimensional conical subregions in the interior of the backward facing ray cone that is formed by taking the convex hull of the backward (negative $x_1$ direction) characteristics (the characteristic $C_i$ is the line in the direction of $\alpha_{xi}$) issuing from $P$. In order to describe more exactly these conical subregions of the backward ray cone it is convenient to introduce the concept of a wedge. A wedge is simply the planar area between two backward characteristics issuing from $P$. The wedge formed by the backward characteristics $C_p$ and $C_q$ issuing from $P$ is denoted $\omega_{pq}$. Sometimes $\omega_{pq}$ is used to denote only that part of $\omega_{pq}$ between $P$ and the initial hyperplane; the exact meaning of $\omega_{pq}$ being clear from the context. The wedges generated by every pair of backward characteristics issuing from $P$ form the sides of the backward ray cone and divide it into the subcones $D^i$. Two points lie in the same subcone if the line segment connecting them does not intersect a wedge. Sometimes $D^i$ is also used to denote only that part of the subcone between $P$ and $x_1 = 0$.

For an arbitrary component $U^K$ of $U$ we will define in each subcone $D^i$ a solution, $W^i$, of $L^* = 0$ and together these solutions will comprise the set of R.F. for the component $U^K$ of $U$. Throughout the paper capital $K$ denotes the index of the arbitrary component of $U$ for which R.F. are being defined.

Cauchy data for the R.F. is defined on the wedges that form the boundaries of the domains $D^i$. The motivation for the specification of this Cauchy data is explained in the next section.

3. Specification of the Cauchy data. The value of $U^K$ at $P$ can be expressed in terms of quadratures of $U$ and auxiliary functions over sections of the wedges and the initial hyperplane. Thus, by employing Green's identity

\begin{equation}
\int_{C_K} V(D_K U^K - a_{KK} U^K) \, ds + \int_{C_K} U^K (D_K V + a_{KK} V) \, ds = VU^K \bigg|_{P_K}'
\end{equation}

In equation (3.1), $P_K'$ is the point where the characteristic $C_K$ intersects the initial data plane and $V$ is an, as yet, unspecified function. When the $K$th equation of (1.1) is substituted into (3.1) and $V$ is chosen to be the solution of

\begin{equation}
D_K V + a_{KK} V = 0
\end{equation}

and

\begin{equation}
V(P) = 1
\end{equation}