ON THE "ZERO-TWO" LAW*

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ABSTRACT
Results of Ornstein-Sucheston, are extended to non-separable measure spaces and operators that are not induced by a transition probability.

Notation. We shall use the notation of [2]. Let \((X, \Sigma, m, P)\) be a Markov process with \(m(X) = 1\). Assume that \(P\) is ergodic and conservative: If \(0 \leq f \in L_\infty\) and \(f \neq 0\) then \(\sum_{n=0}^\infty P^n f = \infty\). Note that all inequalities employed are a.e. inequalities.

1. The sequence of suprema

Let us define:
\[ h_n = \sup\{(P^n g - P^{n+1} g) : -1 < g < 1\}. \]

Note that the supremum is in the \(L_\infty\) sense as defined in [1, IV.11.7].

**THEOREM 1.1.** The sequence \(h_n\) satisfies:
(a) \(0 \leq h_n \leq 2\)
(b) \(h_n \geq h_{n+1}\)
(c) \(P h_n \geq h_{n+1}\)
(d) \(\lim h_n = \text{Const.}\)

**PROOF.**
(a) \(0 = P^n 0 - P^{n+1} 0 \leq \sup\{(P^n g - P^{n+1} g) : -1 \leq g < 1\} \leq P^n 1 + P^{n+1} 1 \leq 2\).
(b) \(P^{n+1} g - P^{n+2} g = P^n(P g) - P^{n+1}(P g) \leq h_n\) since \(-1 \leq P g \leq 1\) if \(-1 \leq g \leq 1\).
(c) For every \(-1 \leq g \leq 1\)
\[ P h_n \geq P(P^n g - P^{n+1} g) = P^{n+1} g - P^{n+2} g \]
and the supremum on the right hand side is \(h_{n+1}\).

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(d) Let \( h = \lim h_n \), then \( 0 \leq h \leq 2 \). Now \( Ph = \lim Ph_n = \lim h_{n+1} = h \) and by [2, chap. II, th. B] \( h = \text{Const.} \).

Let us denote the above constant by \( \alpha \).

**COROLLARY.** \( 0 \leq \alpha \leq 2 \) and \( \alpha = 2 \) if and only if \( h_n(x) = 2 \) for all \( x \) and \( n \).

As in [2, p. 54], we define \( U_n = \inf\{P^n, P^{n+1}\} \) by

\[
0 \leq f \in L_\infty: U_n f = \inf\{P^{n+1}f' + P^n(f-f'): 0 \leq f' \leq f\}.
\]

**THEOREM 1.2.** \( U_n 1 = 1 - \frac{1}{2}h_n \).

**PROOF.** \( f = \frac{1}{2}(g + 1) \) sends the family \( -1 \leq g \leq 1 \) onto \( 0 \leq f \leq 1 \). Thus \( P^n f - P^{n+1} f = \frac{1}{2}(P^n g - P^{n+1} g) \) or

\[
U_n 1 = \inf\{1 - (P^n f - P^{n+1} f): 0 \leq f \leq 1\} = 1 - \sup\{P^n f - P^{n+1} f: 0 \leq f \leq 1\} = 1 - \frac{1}{2}h_n.
\]

**COROLLARY.** \( \alpha < 2 \) if and only if there exists a Markov operator \( Q \neq 0 \) and an integer \( n \) such that \( Q \leq P^n \) and \( Q \leq P^{n+1} \).

2. If \( \alpha < 2 \) then \( x = 0 \). Throughout this section we will assume that \( \alpha < 2 \). For every integer \( k \) the operator \( P^k \) is conservative; see [3, cor. 2]. On the other hand \( P^k \) need not be ergodic. Put \( \Sigma_i(P^k) = \{A: P^k 1_A = 1_A\} \).

**LEMMA 2.1.** The \( \sigma \)-field \( \Sigma_i(P^k) \) is atomic. If \( A \in \Sigma_i(P^k) \) is an atom then \( P^i 1_A 0 \leq i < k \) are characteristic functions of disjoint sets.

**PROOF.** If \( \Sigma_i(P^k) \) is non-atomic, let \( A_n \) be a decreasing sequence of sets in \( \Sigma_i(P^k) \) with \( m(A_n) \rightarrow 0 \). Now

\[
0 = (I - P^k) 1_A = (I - P)(1_A + P 1_A + \ldots + P^{k-1} 1_A).
\]

Thus \( 1_A + P 1_A + \ldots + P^{k-1} 1_A = \text{Const.} \) as \( P \) is ergodic, or

\[
1_A + P 1_A + \ldots + P^{k-1} 1_A \geq 1.
\]

But as \( n \rightarrow \infty \) each term tends to zero. Let now \( A \) be an atom of \( \Sigma_i(P^k) \) then \( P^i 1_A \) are characteristic functions, by [2, chap. III, th. A], of the sets \( A_i \) and \( A_i \cap A_j \in \Sigma_i(P^k) \) too. Hence, the intersection is empty.

A more general result is obtained in [4, th. 1].

**COROLLARY.** If \( \alpha < 2 \) then \( P^k \) is ergodic for every integer \( k \).

**PROOF.** Let \( A \) be as above, then \( h_{nk} \geq 2(P^{nk} 1_A - P^{nk+1} 1_A) = 2 \) on \( A \) and \( \alpha = 2 \), too.

In the following construction we use the methods of [5].