ON GENERALIZED ABSOLUTELY MONOTONE FUNCTIONS

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ABSTRACT
Sufficient conditions for generalized absolutely monotone functions to possess a Taylor-type expansion in terms of the corresponding Extended Tchebycheff systems were found by Karlin and Ziegler. The question of necessary conditions, however, was left open. In this paper we solve this question by finding necessary and sufficient conditions for the validity of the expansion. The structure of the cone of generalized absolutely monotone functions and its extreme rays are also discussed.

We start by recalling briefly some definitions and basic results which will be used in the sequel. For a more detailed discussion of the results quoted here the reader is referred to the first paper on the topic [2] and to the monograph by Karlin and Studden [1].

Let \( \{u_i(t)\}_{i=0}^{\infty} \) be an infinite sequence of functions belonging to \( C^\infty[a, b] \) and such that for all \( n, n = 0, 1, \ldots, \{u_0, u_1, \ldots, u_n\} \) constitutes an Extended Tchebycheff system on \([a, b]\). With no loss of generality we may assume that the \( u_i \)'s are of the form \( u_i(t) = \phi_i(t; a) \) where

\[
\phi_i(t; x) = \begin{cases} 
\int_a^t \left( \prod_{j=0}^{i-1} w_j(\xi_j) \right) \int_a^{\xi_1} w_{i-1}(\xi_{i-1}) \cdots \int_a^{\xi_1} w_0(\xi_0) \, d\xi_{i-1} \cdots d\xi_1 & x \leq t \leq b \\
0 & a \leq t < x
\end{cases}
\]

for \( i = 0, 1, \ldots \) \( x \in [a, b] \)

and \( \{w_k(t)\}_{k=0}^{\infty} \) is a sequence of positive functions, each of class \( C^\infty[a, b] \). With these functions we associate the sequence of first order differential operators

\[
D_i f(t) = \frac{d}{dt} \left( \frac{1}{w_i(t)} \right) f(t), \quad i = 0, 1, \ldots
\]

and the \( k + 1 \)-st order differential operators

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L_{-1}f(t) = f(t), \quad L_kf(t) = (D_kD_{k-1} \cdots D_0)f(t) \quad k = 0, 1, \ldots

**Definition 1.** A function \( \phi(t) \) defined on \((a, b)\) is called "generalized absolutely monotone" (abbreviated G.A.M.) with respect to \( \{u_i(t)\}_{i=0}^{\infty} \) provided \( \phi(t) \) is of class \( C^\infty(a, b) \) and satisfies the inequalities

\[
\phi(t) \geq 0, \quad L_k\phi(t) \geq 0 \quad \text{for all } t \in (a, b),
\]

The concept of "generalized absolute monotonicity" is intimately connected with the concept of "generalized convexity cones".

**Definition 2.** A function \( \psi(x) \) belongs to \( P(u_0, \ldots, u_n) \) (and is called "convex with respect to \((u_0, \ldots, u_n)\)" if for every set of points \( x_0 < x_1 < \ldots < x_{n+2} < b \)

the determinant inequality

\[
\begin{vmatrix}
  u_0(x_1) & \cdots & u_0(x_{n+2}) \\
  u_1(x_1) & \cdots & u_1(x_{n+2}) \\
  \vdots & \ddots & \vdots \\
  u_n(x_1) & \cdots & u_n(x_{n+2}) \\
  \psi(x_1) & \cdots & \psi(x_{n+2})
\end{vmatrix} \geq 0
\]

prevails.

It is proved in [2] that the cone of G.A.M. functions coincides with the intersection cone

\[ P_A = P^+ \cap \left[ \bigcap_{n=0}^{\infty} P(u_0, \ldots, u_n) \right] \]

where \( P^+ \) denotes the cone of continuous non-negative functions defined on \((a, b)\). It is also shown that, if \( f(x) \) is a G.A.M. function, then for all \( n, n = 0, 1, \ldots \) the following Taylor-type formula holds:

\[
f(t) = \int_a^b \phi_n(t; x)L_nf(x)dx + \sum_{k=0}^{n} \frac{L_{k-1}f(a^+)}{w_k(a)} u_k(t).
\]

Having all these facts at our disposal, we are prepared to state the first major theorem. We note that with no loss of generality we may take \( a = 0, \ b = 1 \).

**Theorem 1.** If the sequence \( u_i(t), \ i = 0, 1, \ldots, \) generates the totality of the extreme rays of the cone \( P_A \), then the expansion

\[
\text{(5)}
\]